So far we have discussed how a given perturbation evolves with time. In order to describe the cosmic structures, we need to know the properties of the cosmological perturbations.
Linear power spectra

Since we can write $\delta(k,t) \propto T(k,t)\delta(k,t_i)$, the power spectrum can be written as

$$P(k,t) \equiv \langle |\delta(k,t)|^2 \rangle \propto T^2(k,t)P(k,t_i),$$

Thus if the initial power spectrum $P(k,t_i)$ and the transfer function are known, we can obtain the power spectrum at any subsequent time.

The power spectrum is the most important property of the cosmic density field. If the cosmic density field is Gaussian, the power spectrum completely specifies the statistical properties of the cosmic density field.
The initial power spectrum

The initial power spectrum is assumed to have the form:

\[ P(k, t_i) = A_i k^n \quad (n = 1), \]

which is consistent with the prediction of inflation theory and CMB observations.

Since inflation theory is not complete, the amplitude \( A_i \) has to be determined through observations. Note that to determine \( A_i \), we only need to specify \( P(k, t_i) \) for any given \( k \). There are therefore many ways to normalize the power spectrum.

e.g. (1) by CMB observations (e.g. COBE); (2) by cluster abundance
For $P(k)$ with a given shape, the amplitude is fixed if we know the value of $P(k)$ at any $k$, or the value of any statistic that depends only on $P(k)$.

Normalization is usually represented by the value of $\sigma_8 \equiv \sigma(8h^{-1}\text{Mpc})$, where

$$\sigma^2(R) = \frac{1}{2\pi^2} \int k^3 P(k) \frac{9[\sin(kR) - kR\cos(kR)]^2}{(kR)^6} \frac{dk}{k},$$

where $P(k)$ is the linear power spectrum at the present time $t_0$.

Reason: the variance of counts-in-cells for normal galaxies is about unity in spheres of radii $R = 8h^{-1}\text{Mpc}$.

If we write $\delta_{\text{gal}} = b\delta$ with $b = \text{constant}$, then $\sigma(8h^{-1}\text{Mpc}) = \sigma_{\text{gal}}(8h^{-1}\text{Mpc})/b \approx 1/b$, where $b$ is a bias parameter.

$$\delta(x; R) = \int \delta(y) W(x - y; R) d^3y,$$

$$\delta_k(R) = \delta_k W(k; R),$$

$$\sigma^2(R) = (1/V) \int \delta^2(x, R) d^3x = \int d^3k \delta_k^2 W^2(k; R)$$
The values of $\sigma_8$ in different cosmogonies

<table>
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<th>$\Omega_{\nu,0}$</th>
<th>$\Omega_{\Lambda,0}$</th>
<th>$h$</th>
<th>$\sigma_8$(COBE)</th>
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</table>
The power spectra for various cosmogonic models. The initial power spectrum is assumed to be scale invariant (i.e. $n = 1$), and the spectra are normalized to reproduce the COBE observations of CMB anisotropies.
Cosmic density field

For a given cosmology, the density field at a cosmic time $t$, is described by

$$\delta(x, t) \quad \text{or} \quad \delta_k(t).$$

How to specify a linear density field? to specify $\delta(x)$ for all $x$ or to specify $\delta_k$ for all $k$? NO!

- We consider the cosmic density field to be the realization of a random process, which is described by a probability distribution function:

$$\mathcal{P}_x(\delta_1, \delta_2, \ldots, \delta_N) \, d\delta_1 \, d\delta_2 \cdots \, d\delta_N, \quad (N \to \infty)$$

Thus, we emphasize the properties of $\mathcal{P}_x$, rather than the exact form of $\delta(x)$. 
• The form of \( P_x(\delta_1, \delta_2, \cdots, \delta_N) \): is determined if we know all of its moments:

\[
\left\langle \delta_1^{\ell_1} \delta_2^{\ell_2} \cdots \delta_N^{\ell_N} \right\rangle \equiv \int \delta_1^{\ell_1} \delta_2^{\ell_2} \cdots \delta_N^{\ell_N} P_x(\delta_1, \delta_2, \cdots, \delta_N) \, d\delta_1 \, d\delta_2 \cdots \, d\delta_N,
\]

where \((\ell_1, \ell_2, \cdots, \ell_N) = 0, 1, 2, \cdots\).

In real space:

\[
\langle \delta(x) \rangle = 0, \quad \xi(x) = \langle \delta, \delta \rangle, \quad \text{where} \quad x \equiv |x_i - x_j|.
\]

In Fourier space:

\[
\langle \delta_k \rangle = 0, \quad P(k) \equiv V_u \langle |\delta_k|^2 \rangle \equiv V_u \langle \delta_k, \delta_{-k} \rangle = \int \hat{\xi}(x) \exp(i \mathbf{k} \cdot \mathbf{x}) \, d^3 \mathbf{x},
\]

In general, it is quite difficult to describe a random field.
Gaussian Random Fields

- In real space:

\[
P(\delta_1, \delta_2, \cdots, \delta_n) = \frac{\exp(-Q)}{[(2\pi)^n \det(\mathcal{M})]^{1/2}}; \quad Q \equiv \frac{1}{2} \sum_{i,j} \delta_i (\mathcal{M}^{-1})_{ij} \delta_j,
\]

where \( \mathcal{M}_{ij} \equiv \langle \delta_i \delta_j \rangle \). For a homogeneous and isotropic field, all the multivariate distribution functions are invariant under spatial translation and rotation, and so are completely determined by the two-point correlation function \( \xi(x) \)!
In Fourier space:

\[ \delta_k = A_k + iB_k = |\delta_k| \exp(i\varphi_k). \]

Since \( \delta(x) \) is real, we have \( A_k = A_{-k}, \ B_k = -B_{-k} \), and so we need only Fourier modes with \( k \) in the upper half space to specify \( \delta(x) \). It is then easy to prove that, for \( k \) in the upper half space,

\[ \langle A_k A_k' \rangle = \langle B_k B_k' \rangle = \frac{1}{2} V_u^{-1} P(k) \delta^{(D)}_{kk'}; \quad \langle A_k B_k' \rangle = 0, \]

Thus As a result, the multivariate distribution functions of \( A_k \) and \( B_k \) are factorized according to \( k \), each factor being a Gaussian:

\[ P(\alpha_k) \, d\alpha_k = \frac{1}{\left[ \pi V_u^{-1} P(k) \right]^{1/2}} \exp \left[ -\frac{\alpha_k^2}{V_u^{-1} P(k)} \right] \, d\alpha_k, \]
In terms of $|\delta_k|$ and $\varphi_k$, the distribution function for each mode, $\mathcal{P}(A_k)\mathcal{P}(B_k)\, dA_k\, dB_k$, can be written as

$$\mathcal{P}(|\delta_k|, \varphi_k) \, |\delta_k| \, d\varphi_k = \exp \left[ -\frac{|\delta_k|^2}{2V^{-1}_u P(k)} \right] \frac{|\delta_k| \, d|\delta_k| \, d\varphi_k}{V^{-1}_u P(k) \, 2\pi}.$$ 

Thus, for a Gaussian field, different Fourier modes are mutually independent, so are the real and imaginary parts of individual modes. This, in turn, implies that the phases $\varphi_k$ of different modes are mutually independent and have random distribution over the interval between 0 and $2\pi$.

$P(k)$ is the only function we need!
Is the cosmic density field a Gaussian field?

Gaussian field is the first choice, because

- It is the simplest
- It is quite general because of central-limit theorem
- There is no observational contradiction
- It is predicted by inflation
Realizations of Cosmological Density Field

Particularly simple for a Gaussian field.

- Start with an array of Fourier modes (each is characterized by $k$)
- Assign each mode a random amplitude $|\delta_k|$ and a random phase $\varphi_k$ according to the distribution function.
- Obtain the density perturbation field, $\delta(x)$, via the Fast Fourier Transform (FFT).

Clearly, the resulting perturbation fields differ in different realizations!
Gravitational Clustering

A cosmological density field is generally unstable. Because of gravitational interaction, regions with initial density above the mean grow more overdense in the passage of time, while underdense regions tend to grow more underdense. This kind of gravitational instability in the linear regime has been described in detail.

In the linear regime, the perturbations in different Fourier mode evolve independently, and so gravitational instability enhance the contrast of the fluctuations in a cosmological density field but does not change the correlations between different modes.

In the nonlinear regime, the evolution of the density field is generally complicated, because the equations of motion are nonlinear, and it is in general very difficult to come up with accurate analytical description. Because of this, the exact evolution of the density field is usually solved by means of N-body simulations.
Hierarchical Clustering

In the linear regime, the density perturbation $\delta(x,t)$ grows with time as $\delta(x,t) \propto D(t)$, and so the power variance. $\sigma^2(R;t) \propto D^2(t)$. For $P(k) \propto k^n$, $\sigma^2(R) \propto R^{-(n+3)}$, and so

$$\sigma^2(R;t) = \left[\frac{R}{R_*(t)}\right]^{-(n+3)} = \left[\frac{M}{M_*(t)}\right]^{-(n+3)/3},$$

where

$$M_*(t) \propto [D(t)]^{6/(n+3)}, \quad R_*(t) \propto [D(t)]^{2/(n+3)},$$

are the fiducial mass and length scales on which $\sigma = 1$ at time $t$. 

Nonlinear structures with mass $\sim M$ begin to form in abundance when $\sigma(M; t) \sim 1$. The $t$-dependence of $M_\star(t)$ can be used to understand how structures develop with time.

Since $D(t)$ increases with $t$, the mass scale for nonlinearity, $M_\star(t)$, increases with $t$ for $n > -3$. In this case, structure formation proceeds in a “bottom-up” fashion. On the other hand, for models where the effective spectral index is smaller than $-3$, the first structures to collapse are large pancakes, and smaller structures will then have to form from the fragmentation of larger structures, giving rise to a “top-down” scenario of structure formation.
N-Body Simulations
The answer to this question is still open.

Broadly speaking, two classes of models have been proposed. One is based on inflation models and the other based on topological defects produced during phase transitions in the early universe.

Roughly speaking, inflation models predict isentropic, Gaussian perturbations with spectra close to the Harrison-Zel’dovich spectrum, while defect models generally predict non-Gaussian perturbations.

The inflationary scenario seems to be quite successful
The arguments consist of the following four important points.

1. Since the universe is assumed to have gone through a phase of very fast expansion (inflation), the structures today all have sizes smaller than the horizon size during inflation, and so inflation provides the conditions for generating the initial perturbations in a causal way.

2. If the scalar field driving inflation has negligible self-coupling (as is assumed in most models), different modes in the quantum fluctuations of the field are independent of each other, the density perturbations generated are expected to be Gaussian.

3. The perturbations produced during the inflation phase are those in the energy density of the scalar field. When the inflation is over and as the energy...
density in the scalar field is converted into photons and other particles, we do not expect any segregation between photons and other particles. The perturbations are therefore expected to be isentropic.

4. Since the space during inflation (which is a de Sitter space) is invariant under time translation, the perturbations generated by inflation are expected to be scale-invariant.
Consider two perturbations: generation times, \( t_1 \) and \( t_2 \). Since the same at generation, we have at some late time \( t \),

\[
\frac{\lambda_1}{\lambda_2} = \frac{\exp(Ht)/\exp(Ht_1)}{\exp(Ht)/\exp(Ht_2)} = e^{H(t_2-t_1)}
\]

Time at horizon crossing:

\[
\frac{\lambda_1 a[t_H(\lambda_1)]}{a(t)} = ct_H(\lambda_1) \sim \frac{c}{H[t_H(\lambda_1)]} \sim \frac{c}{H[t_H(\lambda_2)]} \sim \frac{\lambda_2 a[t_H(\lambda_2)]}{a(t)}
\]

Thus, \( \lambda_1/\lambda_2 = e^{H[t_H(\lambda_2)-t_H(\lambda_1)]} \) Hence \( t_H(\lambda_2) - t_2 \sim t_H(\lambda_1) - t_1 \). Amplitudes must be the same at horizon crossing, scale free!
Zeldovich spectrum

\[ P(k) \propto k^n \quad (n = 1) \]

\[ \Delta^2 \propto k^3 P(k) \propto k^{3+n} \]

Since \( \phi_k \propto \delta_k/k^2 \),

\[ \Delta^2_\phi \propto \Delta^2/k^4 \propto k^{n-1} \]

which is independent of \( k \) if \( n = 1 \).

For superhorizon perturbation, \( \delta \propto 1/\rho a^2 \propto t^{2/3} \)

At horizon crossing \( ra(t_H) = ct_H \), and so \( t_H \propto r^3 \).

Thus,

\[ \Delta(t_H) \propto r^{-(n+3)/2} t_H^{2/3} \propto r^{-(n+3)/2} r^2 = r^{(1-n)/2} \]