Spiral galaxies

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- Basic properties of disk galaxies
- Formation of galaxy disks
Exponential Disks

Except in the inner part (where bulge component may be important), the surface-brightness distribution can well be approximated by an exponential profile:

\[ I(R) = I_0 \exp \left( -\frac{R}{R_d} \right), \quad I_0 = \frac{L_d}{2\pi R_d^2}, \]

\( R \): cylindrical radius, \( R_d \): the exponential scalelength, \( I_0 \): central luminosity surface density, and \( L_d \): disk total luminosity.

Milky Way: \( R_d \approx 3.5 \text{kpc} \), and \( I_0 \approx 150 \text{L}_\odot \text{pc}^{-2} \) (in the V-band).
Many spiral galaxies contain central bulges in addition to the disk components. Bulges have smooth light distribution and many of them have $R^{1/4}$ light profiles. So a simple decomposition is:

$$I(R) = I_0 \exp \left(-\frac{R}{R_d}\right) + I_{\text{eff}} \exp \left\{-7.67 \left(\frac{R}{R_{\text{eff}}}\right)^{1/4} - 1\right\},$$

where $R_{\text{eff}}$ is the effective (half-light) radius of the bulge.

The bulge/total ratio (in luminosity) is

$$B/T = \frac{R_{\text{eff}}^2 I_{\text{eff}}}{R_{\text{eff}}^2 I_{\text{eff}} + 0.28 R_d^2 I_0},$$

independent of distance.
Uncertainties

There are uncertainties in this procedure.

- Some bulges may not be fit well by the $R^{1/4}$ profile. This problem may be partly overcome by using the Sersic profile,

\[ I(R) = I_0 \exp(-aR^{1/n}), \]

to model the bulge.

- Although a disk/bulge decomposition can always be done for most spirals, it is unclear if such decomposition always makes physical sense.
Disks are not infinitesimally thin but have vertical ($z$-) structure. The luminosity density as a function of $z$ and $R$ can fit reasonably well by

$$
\rho(R, z) = \rho_0(R, 0) \text{sech}^2 \left( \frac{z}{2H_d} \right),
$$

where $H_d$ is called the scale height (assumed to be independent of $R$).

For our own Galaxy, the scaleheight of the young (thin) disk is $\sim 0.3 \text{kpc}$, while that for the old (thick) disk can be as large as $\sim 1.3 \text{kpc}$. 
Galaxy bars are bar-like structures in $\sim 1/2$ of the spiral population. Appear as isophotes squarer than ellipses: $(|x|/a)^c + (|y|/b)^c = 1$, where $a$, $b$ and $c$ are constants, with larger $c$ corresponding to squarer isophote.
Spiral Arms

Spiral galaxies show variety of spiral structures: Some are grand-design arms (usually two); others are arm segments.

How to quantify the spiral arm structure? The center of a spiral arm is a mathematical curve in the $R-\phi$ plane, $\phi + g(R) = \text{constant}$, $R$ radius from center, $\phi$ azimuthal angle. For $m$ identical arms, spiral pattern is invariant under a rotation of $2\pi/m$, the spiral curve is a wave with phase given by

$$m\phi + f(R) = \text{constant} \pmod{2\pi},$$

$f(R) = mg(R)$: shape function. Thus spiral arm structure can be expanded as

$$I(R, \phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dp I_m(p) e^{i(pu + m\phi)},$$

Here $u \propto f(R)$. The pitch angle: $\psi \equiv -\arctan(m/p)$ ($\tan(\psi) = du/d\phi$). For $m'$-armed spiral with pitch angle $\psi'$, $I_m(p)$ peaks at $m = m'$, $p = -m/\tan(\psi')$. 
Spiral galaxies contain neutral hydrogen (HI) and molecular hydrogen ($\text{H}_2$). These components can be observed as emission from the 21-cm lines of HI and the mm-lines of CO. Gas mass fraction is about 5% in early-type (Sa) galaxies, and about 25% in late-type (Sm) galaxies. In general, the distribution of HI is much more extended than that of $\text{H}_2$. 
Stars and cold gas in galaxy disks are rotating around the axes of the disks. The rotation is described by its rotation curve \( V_{\text{rot}}(R) \). Disk rotation curves can be measured (by long-slit spectroscopies) either at the optical wavelengths from emission lines (e.g. H\(\alpha\)) in HII regions, or at radio wavelengths from the 21-cm emission line from neutral (HI) gas.
The rotation curve is a direct measure of the gravitational force (and mass). Assuming spherical symmetry, the total mass within $r$ is

$$M(r) = rV_{\text{rot}}^2(r)/G.$$  \hspace{1cm} (1)

In the outer region where $V_{\text{rot}}(r)$ is roughly a constant, $M(r) \propto r$.

Thus, the total mass depends significantly on how far the flat rotation curve extends.
Scaling Relation

The Tully-Fisher relation in the $I$-band. $W$ is the linewidth of the HI 21 cm line and is about twice $V_{\text{max}}$. [Adopted from Giovanelli et al.]
Disk Angular Momentum

Disks are supported by rotation. The specific angular momentum of disk material at $R$ is

$$J_d(R) = V_c(R)R,$$

where $V_c(R)$ is the rotation curve.

In the outer part where the rotation curve is flat, $J_d(R) \propto R$; while for the part where $V_c(R)R$, $J_d(R) \propto R^2$.

Assuming that disk material within $R$ has specific angular momentum less than $J_d(R)$, then

$$M_d(< J_d) = M_d(R).$$

The total angular momentum of the disk is

$$J_d = 2\pi \int_0^\infty V_c(R)\Sigma(R)R^2 dR.$$
For a flat rotation curve $V_c(R) = V_c$,

$$M_d(< J_d) = M_d \left[ 1 - \left( 1 + \frac{2M_d J_d}{J_d} \right) \exp \left( -\frac{2M_d J_d}{J_d} \right) \right],$$

where $J_d = 2M_d R_d V_c$ is the total angular momentum of an exponential disk with flat rotation curve.

The importance of rotation is given by the spin parameter,

$$\lambda = J|E|^{1/2}/GM^{5/2}.$$

where $M$, $J$, and $E$ are the mass, angular momentum, and total energy of the system. For an isolated exponential disk, the total angular momentum is

$$J_d = 2\pi \int_0^\infty V_c(R) \Sigma(R) R^2 dR \approx 1.11G^{1/2}M_d^{3/2}R_d^{1/2}.$$
Applied to an isolated exponential disk the virial theorem gives $E = -T$,

$$E = -2\pi \int_0^\infty \frac{V_c^2(R)}{2} \Sigma(R) R dR \approx -5.8G\Sigma_0^2R_d^3.$$  

The spin is then

$$\lambda \approx 0.425 \text{ for an isolated exponential disk.}$$
The Formation of Galaxy Disks

General Discussion

A natural way to form a disk is through dissipational collapse of gas cloud with some initial angular momentum.

Effective cooling of a gas cloud causes it to approach a state with energy as low as possible while conserving angular momentum.

The preferred state is a rotating thin disk.
Monolithic Collapse or Continuous Accretion?

How does a protogalactic cloud collapses to form a gaseous disk? Two scenarios have been proposed.

(1) Eggen, Lynden-Bell & Sandage (1962): the formation of a disk galaxy involves a monolithic collapse of a cold, nearly spherical cloud with some initial angular momentum. The cloud is initially in free-fall under self-gravity, and as it shrinks it spins up to conserve angular momentum. The defining property of this model is the short timescale for the assembly of a galaxy disk from protogalactic gas. Thus, each disk has a well-defined time of formation in this scenario!

(2) Searle (1977): Galaxy disks are assumed to be built up by continuous accretion and merger of gas clumps. As the accreted gas flows inwards, it dissipates and is spin up, and eventually adds to an existing gaseous disk.

These two scenarios merely point out two kinds of initial conditions that can lead to the formation of disks. Neither of them provides an explanation for the initial conditions required.

In hierarchical cosmogonies (e.g. CDM), the formation of galaxies is basically a process of accretion and merging. An important new aspect: galaxies form in dark halos.
Non-Self-Gravitating Disks in Isothermal Spheres

The density profile, radius and mass of an isothermal sphere:

\[ \rho(r) = \frac{V_c^2}{4\pi Gr^2}, \quad r_h = \frac{V_c}{10H(z)}, \quad M = \frac{V_c^3}{10GH(z)}, \]

\( V_c \): circular velocity of the halo, \( H(z) \): Hubble constant at redshift \( z \).

Disk mass:

\[ M_d = m_dM \approx 1.8 \times 10^{11} h^{-1} M_\odot \left( \frac{m_d}{0.05} \right) \left( \frac{V_c}{250 \text{ km s}^{-1}} \right)^3 \left[ \frac{H(z)}{H_0} \right]^{-1}. \]

Neglecting disk gravity, \( V_c(R) = V_c \) and disk angular momentum is

\[ J_d = 2\pi \int V_c \Sigma(R) R^2 dR = 2M_d R_d V_c, \]

where \( R_d \) is the scalelength for an expotential disk.
If we write \( J_d = j_d J_h \) and relate \( J_h \) to the spin parameter \( \lambda \) of the halo, then

\[
R_d = \frac{\lambda GM^{3/2}}{2V_c|E_h|^{1/2}} \left( \frac{j_d}{m_d} \right).
\]

From virial theorem, \( E_h = -GM^2/(2r_h) = -MV_c^2/2 \). Thus

\[
R_d = \frac{1}{\sqrt{2}} \left( \frac{j_d}{m_d} \right) \lambda r_h \approx 8.8h^{-1}\text{kpc} \left( \frac{\lambda}{0.05} \right) \left( \frac{V_c}{250\text{km}\text{s}^{-1}} \right) \left[ \frac{H}{H_0} \right]^{-1} \left( \frac{j_d}{m_d} \right),
\]

\[
\Sigma_0 \approx 380h \left( \frac{m_d}{0.05} \right) \left( \frac{\lambda}{0.05} \right)^{-2} \left( \frac{V_c}{250\text{km}\text{s}^{-1}} \right) \left[ \frac{H(z)}{H_0} \right] \left( \frac{m_d}{j_d} \right)^2 \text{M}_\odot\text{pc}^{-2}.
\]

For a ‘typical’ halo with \( \lambda = 0.05 \) and \( V_c = 200\text{km}\text{s}^{-1} \) the \( R_d = 7[H_0/H(z)]h^{-1}\text{kpc} \), or \( R_d \approx 7h^{-1}\text{kpc} \) for assembly near \( z = 0 \) and \( R_d \approx 3h^{-1}\text{kpc} \) for assembly near \( z = 1 \).
As a disk forms in the center of a dark halo, the halo structure is modified due to the gravity of the disk. This, in turn, changes the disk rotation curve. Here consider a simple model under some strict assumptions.

If the growth of the disk in its halo is so slow that the potential of the system changes only little during a typical orbital time of an dark matter particle, the system may be considered static during an orbital time. Such slow variations of potential are called adiabatic.

In an adiabatic process, the action integral of a particle

\[ \int p_i \, dq_i \]

is conserved, where \((q_i, p_i)\) being the pair of canonical coordinates and their conjugate momenta and the integral is taken over a single orbit.
For a spherically symmetric potential and for particles in nearly circular orbit:

\[ GM_f(r_f)r_f = GM(r_i)r_i. \]

The final mass is the sum of the dark matter mass inside the initial radius \( r_i \) and the mass contributed by the disk:

\[ M_f(r_f) = M_d(r_f) + M_h(r_i)(1 - m_d). \]

For given \( M_d(r) \) and \( M(r) \), the above two equations can be used to solve \( r_f \) as a function of \( r_i \), thereby giving the final total density profile \( M_f(r_f) \) and the modified halo density profile \( M_f(r_f) - M_d(r_f) \).
More realistic models of disk formation should include (1) realistic halo density profiles and (2) disk self-gravity.

Consider a halo with some unperturbed density profile:

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dM(r)}{dr},$$

where $M(r)$ is the halo mass within radius $r$. Assume baryons ending up in the disk initially had the same density profile and specific angular momentum as the dark matter. Write

$$E = -\frac{GM^2}{2r_{200}} F_E,$$

where $F_E$ is a factor which depends on the exact form of $\rho(r)$. 
For a given rotation curve, $V_c(R)$, the total angular momentum of the disk is

$$J_d = \int_0^{r_h} V_c(R) R \Sigma(R) 2\pi R dR = M_d R_d V_h \int_0^{r_h/R_d} e^{-u^2} \frac{V_c(Rdu)}{V_h} du$$

where $V_h \equiv V_c(r_h)$ is unaffected by disk formation.

In practice we can set the upper limit of integration to infinity because the disk surface density drops exponentially and $r_{200} \gg R_d$. Then

$$R_d = \frac{1}{\sqrt{2}} \left( \frac{j_d}{m_d} \right) \lambda r_h F_R,$$

$$F_R = 2F_E^{-1/2} \left[ \int_0^\infty e^{-u^2} \frac{V_c(Rdu)}{V_{200}} du \right]^{-1}.$$
Disk formation alters the rotation curve because of disk gravity and of halo contraction induced by the disk. The final $V_c(r)$ may be written as

$$V_c^2(r) = V_{c,d}^2(r) + V_{c,DM}^2(r).$$

For given $M(r)$, $\lambda$, $m_d$ and $j_d$, the above equations can be solved by iteration to yield $R_d$, and $V_c(R)$. Such models with the CDM halo profile has been examined (e.g. Mo, Mao & White 1998). The model predictions are in reasonable agreement with the observed properties of present-day disks, provided:

(i) the masses of disks are a few percent of those of their haloes;
(ii) the specific angular momenta of disks are similar to those of their haloes;
(iii) present-day disks were assembled recently (at $z \ll 1$).
Examples of model rotation curves:
Halos have NFW profile
Disks are exponential
What determines the disk sizes

\[ R_d = \frac{1}{\sqrt{2}} \left( \frac{j_d}{m_d} \right) \lambda r_h F_R, \]

Halo size \( r_h \): for a given \( V_c \), \( r_h \propto 1/H(z) \),
disks are expected to be smaller at higher \( z \).

Spin parameter \( \lambda \): For a given \( V_c \), size has the same distribution as \( \lambda \), i.e. log-normal distribution.
The assembly of a disk galaxy

Suppose dark matter halo has established quasi-equilibrium
Suppose gas accretion rate to the disk is $\dot{M}_d(t)$
Suppose the gas accreted at time $t$ has specific angular momentum distribution $P(J,t)\,dJ$
Suppose disk surface density is $\Sigma(R,t)$ at time $t$

$$2\pi\Sigma(R,t)R\,dR = \dot{M}_d(t)P(J,t)\,dJ$$
where $V(R,t)R = J$, with $V(R,t)$ being the rotation curve at time $t$. 
If gas settles into the disk conserving angular momentum, then

$$\dot{\Sigma}(R,t) = \frac{\dot{M}_d(t)}{2\pi R^2} P(J,t) RV(R,t) \left[ 1 + \frac{\partial \ln V(R,t)}{\partial \ln R} \right]$$

For a given dark matter halo, and $\Sigma(R,t)$, $V(R,t)$ can be obtained, and so the growth of the disk can be solved, once $\dot{M}_d(t)$, and $P(J,t)$ are given.

Specific model: $\dot{M}_d(t)$ determined by radiative cooling

$$P(J,t) = \delta [J - J(r_{\text{cool}}(t))]$$

$J(r)$ is the specific angular momentum distribution in dark matter halo, which can be obtained from $N$-body simulations of dark halos.
The Tully–Fisher Relation

Disk galaxies are observed to cover a large range of $L, R_d, V_c$ and rotation-curve shape, why do they obey a tight Tully-Fisher relation. In the I-band

$$M_I - 5 \log h = -21.00 - 7.68(\log W - 2.5).$$

For a fixed $V_{\text{max}} \approx W/2$ the scatter in $L$ is about 50%.

The tight TF relation implies a close relation between the total gravitational mass and the total amount of stars that can form.

Why?
One might think that this can be achieved if the rotation curve of a disk is dominated by luminous mass. However, this is true only if disks with the same mass have the same surface density distribution.

Consider a self-gravitating exponential disk, \( R \approx 2.16 R_d, V_{\text{max}}^2 \approx 2.5 G \Sigma_0 R_d \). Assuming a disk mass-to-light ratio \( \Upsilon \equiv L_d/M_d \), we have

\[
L_d \approx B \left( \frac{V_{\text{max}}}{200 \text{ km s}^{-1}} \right)^4 \left( \frac{I_0}{100 \mathbf{L}_\odot \text{ pc}^{-2}} \right)^{-1} \left( \frac{\Upsilon_d}{\Upsilon_\odot h} \right)^{-2},
\]

\[
I_0 \equiv \Sigma_0/\Upsilon_d, \quad B = 8.5 \times 10^{11} h^{-2} \mathbf{L}_\odot,
\]

This looks quite like the observed TF relation. There are two problems:
(1) Disk is too massive for a given \( V_{\text{max}} \);
(2) The expected scatter in \( L_d \) for given \( V_{\text{max}} \) is too big, because it is the same as that in \( I_0 (\sim 1 \text{ mag}) \)
Disks in Massive Dark Haloes

As another extreme, we may assume disk gravity to be negligible, and disk rotation curves are determined entirely by dark haloes. This can happen if dark haloes are very massive and concentrated.

The Tully-Fisher relation expected from this model can be obtained from

\[
L_d = A \left( \frac{V_c}{250\text{km s}^{-1}} \right)^\alpha, \quad \alpha = 3, \quad A = 1.7 \times 10^{11} h^{-1} L_\odot \left[ \frac{\Upsilon_d^{-1} m_d}{0.05} \right] \left[ \frac{H(z)}{H_0} \right]^{-1}
\]

This is quite similar to the observed Tully-Fisher relation, provided all the factors in the brackets are of order unity.

Problem: requires a constant \( m_d \sim 0.05 < \Omega_{B,0}/\Omega_0 \)! Not natural.
The Effect of Disk–Halo Interaction

The theory would be much less contrived if the predicted TF relation does not depend very sensitively on the value of $m_d$. This is indeed the case for exponential disks in realistic halos. The interaction between the disk and halo acts to reduce the effect of changing $m_d$ in the TF relation.

Taking into account halo gravity (assumed to be spherically symmetric), we can write

\[ L_d = B \left( \frac{V_{\text{max}}}{200 \text{ km s}^{-1}} \right)^4 \left( \frac{\Upsilon_d}{\Upsilon_\odot h} \right)^{-2} \left( \frac{I_0}{100L_\odot \text{ pc}^{-2}} \right)^{-1} \left( \frac{V_d^2}{V_{\text{max}}^2} \right)^2. \]

\[ \mathcal{F} = \frac{L_{\text{pred}}(V_{\text{max}} = 200 \text{ km s}^{-1})}{L_{\text{obs}}(V_{\text{max}} = 200 \text{ km s}^{-1})} \approx 1.0 \times \left( \frac{\Upsilon_d}{2h} \right)^{-2} \left( \frac{I_0}{300L_\odot \text{ pc}^{-2}} \right)^{-1} \left[ \frac{(V_d/V_{\text{max}})^2}{0.5} \right]^2. \]
Scaling of TF zero-point with \( m_d \):

If \( m_d \rightarrow (1 + \varepsilon)m_d \) but \( R_d = \text{const.} \),
then \( \Sigma_0 \rightarrow (1 + \varepsilon)\Sigma_0 \), \( V_{d}^2 \rightarrow (1 + \varepsilon)V_{d}^2 \), and

\[
\mathcal{F} \rightarrow \mathcal{F}' = \frac{1 + \varepsilon}{(1 + \varepsilon')^2} \mathcal{F}, \quad \varepsilon' = \frac{\varepsilon V_d^2 + \delta V_h^2}{V_{\text{max}}^2}
\]

\( \mathcal{F}' \sim \mathcal{F} (1 + \varepsilon) \) if halo dominated
\( \mathcal{F}' \sim \mathcal{F} / (1 + \varepsilon) \) if disk dominated
\( \mathcal{F}' \sim \mathcal{F} \) if \( V_{d}^2 \sim V_{\text{max}}^2 / 2 \)
The origin of exponential disks

(1) Due to initial angular momentum distribution of disk material:

\[ M_d(< R) = M_i(< J_i) \quad V_c(R)R = J_i, \]

For an singular isothermal sphere, this requires \( J_i(R_i) \propto R_i \)

(2) Due to later evolution: starting with a disk with flat density profile, viscosity of the gas cause angular momentum transfer from inside out. If \( t_{SF} = t_{vis} \), then the final disk is close to exponential.
Disk instabilities

Local instability:

By local instability we mean that the typical size of the density perturbation involved is much smaller than the disk. In this case, the analysis of instability can be simplified.

\[ \Sigma_1(R, \phi, t) = H(R, t) \exp \{i [m\phi + f(R, t)]\}, \]  

where \( f(R, t) \) is the shape function, and \( H(R, t) \) is a slowly varying function of \( R \).

For a fixed \( t \), \( m\phi + f(R, t) = \text{constant} \) defines a curve (on the disk) where the phase of the perturbation mode is the same. The curves defined by

\[ m\phi + f(R, t) = 2n\pi (n = 0, \pm 1, \cdots) \]

correspond to the peaks of the density waves
The radial separation between adjacent arms at a given $\phi$ is $\Delta R$ given by

$$|f(R + \Delta R, t) - f(R, t)| = 2\pi$$

If $\Delta R \ll R$, $f(R + \Delta R(\text{local instability})t) \approx f(R, t) + (\partial f/\partial R)\Delta R$, and so $\partial f/\partial R = 2\pi/\Delta R$. In this case,

$$\Sigma_1(R, \phi, t) \approx \Sigma_a \exp\{ik(R_0, t)(R - R_0)\}, \quad (3)$$

$$\Sigma_a = H(R_0, t)\exp\{i[m\phi_0 + f(R_0, t)]\}, \quad k(R_0, t) \equiv \left[\frac{\partial f(R, t)}{\partial R}\right]_{R_0} = \frac{2\pi}{\Delta R}. \quad (4)$$

Thus, a spiral density wave closely resembles a plane wave in the $R$ direction.

For fluid disk, we can insert the above equation into fluid equations (written in cylindrical coordinates) and obtain the dispersion relation:

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma_0 |k| + k^2 c_s^2, \quad (5)$$
where \( \omega \) is angular frequency, \( \Omega = \frac{V_c}{R} \), \( \kappa \) epicycle frequency

\[
\kappa^2 = 2\frac{V_c}{R} \left( \frac{V_c}{R} + \frac{dV_c}{dR} \right).
\]

For axisymmetric perturbations with \( m = 0 \), \( \omega^2 = \kappa^2 - 2\pi G\Sigma_0|k| + k^2c_s^2 \). Define the following two dimensionless quantities,

\[
Q \equiv \frac{c_s\kappa}{\pi G\Sigma_0}\quad \text{and}\quad \lambda_{\text{crit}} \equiv \frac{4\pi^2 G\Sigma_0}{\kappa^2}.
\]

(6)

\( Q > 1 \) (stable disk), \( Q < 1 \) disk may be unstable
Global Instability

Long-wave perturbations with wavelengths comparable to the disk size can also give rise to disk instability.

Global instability leading to non-axisymmetric (bar-like) structure

Analytic model only possible for special cases. One example Maclaurin disks:

\[ \Sigma_0(R) = \begin{cases} \Sigma_c \left(1 - \frac{R^2}{a^2}\right)^{1/2} & (R \leq a) \\ 0 & (R > a) \end{cases}, \tag{7} \]

where \( a \) is the radius of the disk. The disk is assumed to have a uniform rotation, with angular velocity \( \Omega \).
Perturbation on the disk $\Sigma_1(\xi, \phi)$, where $\xi = (1 - R^2/a^2)^{1/2}$, can be expand in terms of associated Legendre function $P^m_l$. The dispersion relation

$$(\omega - m\Omega)^3 - (\omega - m\Omega) \left\{ 4\Omega^2 + (\ell^2 + \ell - m^2) \left[ \Omega_0^2(1 - g_{\ell m}) - \Omega^2 \right] \right\}$$

$$+ 2m\Omega \left[ \Omega_0^2(1 - g_{\ell m}) - \Omega^2 \right] = 0, \quad (8)$$

where

$$g_{\ell m} = \frac{(\ell + m)! (\ell - m)!}{2^{(2\ell - 1)} \left[ \left( \frac{\ell + m}{2} \right)! \left( \frac{\ell - m}{2} \right)! \right]^2}. \quad (9)$$

$$\Omega_0^2 = \pi^2 G \Sigma_c / 2a. \quad (10)$$
The first modes corresponding to real perturbations of the disk are the $\ell = 2$ modes which have $m = 0$ or 2. The $m = 0$ mode corresponds to axisymmetric expansion (or shrink) and pulsation of the disk.

For $\ell = 2$ and $m = 2$ the dispersion relation reduces to

$$\omega = \Omega \pm \sqrt{\Omega_0^2/2 - \Omega^2}. \quad (11)$$

The mode is dynamically unstable (i.e. $\omega$ has an imaginary part) if

$$\Omega^2 > \Omega_0^2/2. \quad (12)$$

The density perturbation represented by this mode has the form

$$\Sigma_1 = \frac{3R^2}{a\sqrt{a^2 - R^2}} \cos (2\phi - \omega t), \quad (13)$$
which represents a rotating elliptical deformation of the disk.

This mode is sometimes called the bar mode because it deforms the disk into a shape reminiscent of the bars in disk galaxies. The instability is referred to as the **bar instability**.
In the present of dark halo, disk is stabilized. The criterion is

\[ \varepsilon = \frac{V_{\text{max}}}{(GM_d/R_d)^{1/2}} < 1 \]

\[ \varepsilon_m \sim (\lambda/m_d)^{1/2} \]

Galactic bars are believed to be formed through disk bar instability.
Star formation in galactic disks

Disk should be locally unstable to form stars

\[ Q < 1, \quad \Sigma > \Sigma_{\text{crit}} = \frac{c_s \kappa}{\pi G} \]

Empirical relation:

\[ \Sigma_{\text{SFR}} \propto \Sigma_{\text{cold}}^{1.4} \]

Theory:

\[ \dot{\rho} \propto \rho_{\text{cold}}/\tau_{\text{dyn}} \propto \rho_{\text{cold}}^{1/2} \]
The formation of spiral arms

Small material arms: due to shear of local perturbation by differential rotation
Grand-designed arms: spiral density waves or excited by nearby satellites.