Here are a few “walk-throughs” of problems using the notation from class and also showing how to solve the problems algebraically. This should help cement the connections between physical laws, equations, algebra, and how to use proportionalities (instead of a lot of algebra) to solve problems. I will show how to solve problems in a few different ways. I have tried to be very explicit in most of the algebra, and assume that we are all familiar with basic identities.

First, the simplest problem, a direct proportionality. In a direct proportionality, when property A increases to X times as much, property B also increases X times as much. One equation from your sheet with an important direct proportionality is the velocity of an object:

\[
velocity = \frac{dist}{time} = \frac{d}{t}
\]

Here, the velocity is directly proportional to the distance covered. (It’s also inversely proportional to the time elapsed, but forget that for now). We can indicate this by the following notation:

\[v \propto d\]

The symbol \(\propto\) means “proportional to.” The kind of proportionality depends on what the terms look like on the right-hand side. Notice that when we deal with proportionality, we tend to ignore any constant terms in the equations, because they are the same in the two cases we are comparing (you will notice they explicitly divide out in the algebraic solutions). We have isolated the terms that are changing in ways that affect the answer. Right now, we are considering the amount of time to be constant in any comparisons we make. Using proportionalities is most useful for cases where we know that one thing has changed by some factor, and we want to know how that would change something else. We can even calculate a final numerical answer if we know what the parameter started as—for example, if we know we started with “R” equal to “10 blerns” and that at the end, R increases 42 times, at the end R must be equal to 420 blerns. Sometimes the answer will be “42 times,” and sometimes, “420 blerns.”

Returning to velocity, we could imagine a question, then:

A car travels 100 km in 10 seconds. A second car travels 200 km in 10 seconds. Compared to the velocity of the first car, the velocity of the second car is _____ times as fast.

This should seem obvious, but that’s why it’s a good example for exploring the notation.

<table>
<thead>
<tr>
<th>First Car</th>
<th>Second Car</th>
<th>Though Process for each line</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 km</td>
<td>200 km</td>
<td>(\rightarrow) We have actual values for the distances here.</td>
</tr>
<tr>
<td>= (d)</td>
<td>= (2d)</td>
<td>(\rightarrow) If 100 km is (d), then the second car went (2d).</td>
</tr>
<tr>
<td>(\rightarrow)</td>
<td>(\rightarrow) Next, I know that (v \propto d). So…</td>
<td></td>
</tr>
<tr>
<td>(v)</td>
<td>(\rightarrow 2v)</td>
<td>(\rightarrow) the speed of the first car is (v), and (v) changes as (d),</td>
</tr>
<tr>
<td></td>
<td></td>
<td>so the speed of the second car must be (2v).</td>
</tr>
</tbody>
</table>
By now you probably recognize that this is a short cut for the following algebraic solutions:

(1) \[ \frac{v_1}{t_1} = \frac{100 \text{ km}}{1} \quad \text{and} \quad \frac{v_2}{t_2} = \frac{200 \text{ km}}{t_2} \]

I want to calculate

\[ \frac{v_2}{v_1} = \frac{\frac{200 \text{ km}}{t_2}}{\frac{100 \text{ km}}{t_1}} = \frac{200 \text{ km}}{100 \text{ km}} \frac{t_1}{t_2} \quad \text{but} \quad t_1 = t_2 \quad \text{so} \quad \frac{v_2}{v_1} = \frac{200 \text{ km}}{100 \text{ km}} (1) = 2 \]

Note that you could also do even more work by using the 10 seconds:

(2) \[ \frac{v_1}{t_1} = \frac{100 \text{ km}}{10 \text{ s}} = 10 \frac{\text{ km}}{\text{ s}} \quad \text{and} \quad \frac{v_2}{t_2} = \frac{200 \text{ km}}{10 \text{ s}} = 20 \frac{\text{ km}}{\text{ s}} \quad \text{so} \quad \frac{v_2}{v_1} = \frac{20 \frac{\text{ km}}{\text{ s}}}{10 \frac{\text{ km}}{\text{ s}}} = 2 \]

This is a perfectly valid way to do the problem, but subject to errors. If the numbers didn’t work out nicely, you could easily end up with, e.g., 1.9999998 or something, instead of 2. This is more likely to crop up problematically in exercises that have powers or roots. Second, there is a chance of making an algebra or numerical mistake doing all the extra computation. The problem doesn’t ask for any of the actual velocities, so you really don’t need to work them out (in this regard, algebraic approach (1) is better than (2)). However, it is always a good idea to check that you have the right answer by doing the problem some other way, and these are the ways to do it. In fact, you may find it occasionally useful to make up numbers to put into equation to check answers—just be careful you pick numbers that work. For example, if the question had been phrased,

A car travels a distance \( d_1 \) in \( t_1 \) seconds. A second car travels \( 2d_1 \) km in \( t_1 \) seconds. Compared to the velocity of the first car, the velocity of the second car is _____ times as fast.

Here, it might be helpful to use some simple numbers to verify the results that the 2\textsuperscript{nd} car is twice as fast. We could pick \( t_1 \) as 1 second, and \( d_1 \) as 10 meters, but then we have to calculate \( v_1 \) is 10 m/s and make sure that \( d_2 \) is 20 meters and \( v_2 \) is 20 m/s—just making up \( v_1 = 5\text{m/s} \) will not be consistent. It’s also a good idea to take care with using “1” as a substitute value, just because \( 1/1 = 1 \) and \( 1^2=1 \) can obscure more complicated relationships (it should always work, but you might miss something. “2” is often better).

**Inverse-Proportional Relationships**

In class, we have worked the most so far with angular size as an inverse proportionality, so let’s examine an angular size problem, which will demonstrate inverse-linear proportionality (think 1:1 when you hear “linear”).
The full equation (number 5) is: \[
\frac{A^\circ}{57.3^\circ} = \frac{L}{d}
\] where \(L\) is the physical size of the object between the two points you measure the angle (often the diameter in astronomy, but it could be the height or width) and \(d\) is the distance to the object. \(A\) is the angle in degrees from one side of the object to the other (note: it is NOT an area). The 57.3 degrees is essentially a unit; it is what is leftover when you simplify this equation: \[
\frac{A^\circ}{360^\circ} = \frac{L}{2\pi d}.
\]

**Example 1: Angular size.**

You are 100 ft from a train car. Using a protractor and a couple of pencils, you work out that it's an angle of 6 degrees from the bottom of the train car to the top. How much bigger will it be if you move up to 25 ft away?

With algebra, completely solve:

\[
A_{100,6} = \frac{6^\circ}{57.3^\circ} \times 100 \text{ ft} = 10.5 \text{ ft}
\]

\[
A_{25,6} = \frac{L}{d} \quad \Rightarrow \quad \frac{A_{25,6}}{A_{100,6}} = \frac{24.07^\circ}{6^\circ} = 4.0116... \quad \text{Here the degrees cancel in the ratio. We might guess this should be 4...}
\]

Algebra, smarter:

\[
A_{100,6} = \frac{6^\circ}{57.3^\circ} \times 100 \text{ ft}
\]

\[
A_{25,6} = \frac{L}{d} \quad \Rightarrow \quad \frac{A_{25,6}}{A_{100,6}} = \left(\frac{25}{L}ight) \times 57.3 = \left(\frac{1}{25 \text{ ft}}\right) \times \left(\frac{1}{100 \text{ ft}}\right) = 4
\]

Not too bad, but a lot of fractions and canceling.
With the chart:

<table>
<thead>
<tr>
<th>Thought Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before (100ft)</td>
</tr>
<tr>
<td>100 ft</td>
</tr>
<tr>
<td>$d$</td>
</tr>
<tr>
<td>$d = 100$ ft; new $d$ is $25/100 = \frac{1}{4}$ of that.</td>
</tr>
<tr>
<td>$A$</td>
</tr>
<tr>
<td>$A$ goes as $1/d$</td>
</tr>
</tbody>
</table>

The angular size of the train increases to 4 times as big.

**Example 2: Angular size, but a different part of the equation:**

*The Sun and the Moon both subtend an angle of 0.5 degrees on the sky. If the Sun is 400 times larger than the Moon, how much farther away is it?*

Problem solving with proportionality:

\[
\frac{A^\circ}{360^\circ} = \frac{L}{2\pi d} \quad \cdots \quad A \propto \frac{1}{d}
\]

We are usually safe to think about angular size as inversely-proportional to the distance, because very few objects are likely to change angular size. However, if we compare two objects which have different physical sizes, we do have to consider that, at the same distance, a larger $L$ will have a larger $A$. In the question posed, we know that the $A$ as observed is the same for both objects. Algebraically, we are considering:

\[
\frac{A_{\text{Moon}}^\circ}{57.3^\circ} = \frac{L_{\text{Moon}}}{d_{\text{Moon}}} \quad \text{and} \quad \frac{A_{\text{Sun}}^\circ}{57.3^\circ} = \frac{L_{\text{Sun}}}{d_{\text{Sun}}}
\]

with $A_{\text{Sun}} = A_{\text{Moon}} = 0.5^\circ$ and $400L_{\text{Sun}} = L_{\text{Moon}}$

We need to find the ratio of the distances:

\[
\frac{d_{\text{Sun}}}{d_{\text{Moon}}} = \left( \frac{57.3^\circ \times L_{\text{Sun}}}{A_{\text{Sun}}^\circ} \right) \times \left( \frac{A_{\text{Moon}}^\circ}{400L_{\text{Moon}}} \right) = \frac{400(L_{\text{Moon}}/A_{\text{Moon}}^\circ)}{(L_{\text{Moon}}/A_{\text{Moon}}^\circ)} = 400 \times (1) = 400
\]

Notice that I have explicitly factored out the 400 to get two terms that will cancel—in more complex problems, it can help to rewrite terms and pull out a multiplicative constants when trying to get terms to cancel. Notice also that again, it doesn’t actually matter that the size is 0.5 degrees and that information isn’t used.
Using the chart, we would think of the problem this way:

\[ \frac{A^\circ}{57.3^\circ} = \frac{L}{d} \quad \text{rewrite as} \quad d = \frac{L}{A^\circ} \times 57.3^\circ \quad \rightarrow \quad d \propto L \quad \text{since A is constant.} \]

<table>
<thead>
<tr>
<th>Moon</th>
<th>Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>400 L</td>
</tr>
<tr>
<td>( \rightarrow ) 400 d</td>
<td>d goes exactly as L goes</td>
</tr>
</tbody>
</table>

The Sun must be 400 times further away to appear the same size.

Before doing a question, make sure you see how the terms important to solving the problem are related in the equation. Even though angular-size and distance are inverse-proportional, not all questions will be solved by taking 1/change in given terms.

Before moving on, let’s look briefly at Newton’s 2\textsuperscript{nd} Law, \( F = ma \).

You apply a force of 500 Newtons to a 100 kg mass and get an acceleration. How much bigger an acceleration will you get from applying the same force to a 20 kg mass?

\[ F = ma \quad \text{rewrite as} \quad a = \frac{F}{m} \quad \text{so the dependence is} \quad a \propto \frac{1}{m} \quad \text{for constant force.} \]

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 kg</td>
<td>20 kg</td>
</tr>
<tr>
<td>( \frac{1}{5} ) m</td>
<td>( \frac{1}{5} ) m</td>
</tr>
<tr>
<td>change in ( a ) goes as 1/change in ( m ).</td>
<td></td>
</tr>
</tbody>
</table>

The acceleration will be 5 times as much.

**Inverse-Square Laws**

Ok, now it’s time to add some complication. So far, for all of the proportionalities we have looked at, if quantity A changed by X times, quantity B also changed by the same number of times – just sometimes it was multiplied, other times, divided. However, there are many parameters in physical laws where if A changes by X times, B changes by X to a power; and notice that if that happens, then if B changes by X times, A changes by a root of X times. The relationship is not as neatly symmetrical as before. However, using proportionality can still simplify solving problems compared to a complete algebraic approach.
Example from Lecture 08, part 1:

Weight is the force of gravity pulling on you. Assume you weigh 900 Newtons. The radius of the Earth is 6400 km. If you were 12,800 km above the surface of the Earth, you would weigh how many Newtons?

(Enter the number)

The relevant equation is number 11: \[ F = G \frac{m_1 m_2}{d^2} \]

The question is concerned with the change in force \( F \) as a result of the change in distance \( d \). The dependence in the equation is that \( F \) goes as one over the square of the distance, \( d \): \[ F \propto \frac{1}{d^2} \]

From the point of view of the notation in class:

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
<th>Thought Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>6400 km</td>
<td>12,800 km + 6400 km</td>
<td>It’s 12,800 km above the SURFACE, and there’s still 6400 km to Earth’s center.</td>
</tr>
<tr>
<td>( d )</td>
<td>( 3d )</td>
<td>( d = 6400 ) km; ( 3d = 19,200 ) km</td>
</tr>
<tr>
<td>( d^2 )</td>
<td>( (3d)^2 = 9d^2 )</td>
<td>( F ) goes as ( 1/d^2 ) need to calculate square of change of ( d ) change is 9 times for ( d^2 ) change must be 1/9 for ( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( 1/9 F )</td>
<td>1/9 of 900 N is 100 N</td>
</tr>
<tr>
<td>900 N</td>
<td>( \Rightarrow ) 100 N</td>
<td></td>
</tr>
</tbody>
</table>

The force at 12,800 km must be 100 N.

Reality check: we’ve increased the distance between the two objects and have not changed the mass of either. The force of gravity must decrease; our answer shows it has.

Notice in this case, we find a change in distance as a factor of 3, square it for 9, and flip that over into 1/9 for the change in the force. [We’ll compare this in a minute to a similar problem].

Now, for comparison, this is the solution to the problem algebraically:

\[ F_{\text{surface}} = 900 \text{ N} = G \frac{m_{\text{Earth}} m_{\text{person}}}{(6400\text{ km})^2} \]
\[ F_{\text{highup}} = G \frac{m_{\text{Earth}} m_{\text{person}}}{(19,200\text{ km})^2} \]

We need to substitute in the second equation for the masses, which we aren’t given in the problem, and the constant G. We can do this by re-arranging part of the first equation above:

\[ G m_{\text{Earth}} m_{\text{person}} = 900 \text{ N} \times (6400\text{ km})^2 \]
Substituting this into the equation (notice that it is not worth solving out the numbers yet):

\[
F_{\text{highup}} = \frac{G m_{\text{Earth}} m_{\text{person}}}{(19,200 \text{km})^2} = \frac{900N \times (6400 \text{km})^2}{(19,200 \text{km})^2} = \frac{900N \times (6400 \text{km})^2}{(3 \times 6400 \text{km})^2} = \frac{900N \times (6400 \text{km})^2}{(3)^2 \times (6400 \text{km})^2} \]
\[
= \frac{900N}{9} \times (1) = 100N
\]

Notice here that again, in the algebraic solution, it’s easier to factor the numerator and denominator to find like terms to cancel out before punching any numbers in the calculator (like 19200 km squared).

Either the “chart” or the “algebraic” solution gets you to the right answer, and involves essentially the same thought process with respect to the equations. The chart is often a little faster and stresses looking at the way the parameters relate in the equation.

**Example, part 2: Let’s look at the same problem, but if it was asked in reverse:**

Weight is the force of gravity pulling on you. Assume you weigh 900 Newtons. The radius of the Earth is 6400 km. How far from the center of the Earth would you weigh 100 Newtons?

From the point of view of the notation in class:

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
<th>Thought Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>900 N</td>
<td>100 N</td>
<td>Given values</td>
</tr>
<tr>
<td>F</td>
<td>1/9 F</td>
<td>If F=900N, then 1/9 F is 100N</td>
</tr>
<tr>
<td>d^2</td>
<td>9d^2</td>
<td>F goes as 1/d^2</td>
</tr>
<tr>
<td>d</td>
<td>3d</td>
<td>Change in d^2 is 1/change in F; 1/(1/9) is 9.</td>
</tr>
<tr>
<td>6400 km</td>
<td>3x6400 km = 19,200 km</td>
<td>Square root of (9 d^2) is (\sqrt{9} \sqrt{\sqrt{d^2}}) is 3d.</td>
</tr>
<tr>
<td></td>
<td>12,800 km</td>
<td>12,800 km above the surface</td>
</tr>
</tbody>
</table>

Reality check: the force goes down; the answer should be somewhere farther from the Earth’s surface; it is.

**Important**

Notice that in this second example, the change in the given values is a factor of 9 (more precisely, 1/9\(^{th}\)). However, the response IS NOT THE SAME as it was for the first version of the question. In the first example, we took the change and squared it to find the factor to divide the other term by – here, we take the change in the given values, and take the square-root of it to find the change (1/3) to divide (dividing by 1/3 is the same as multiplying by 3) the other parameter by. This is because in the first example, we are looking at \(F \propto \frac{1}{d^2}\) but in the second example, it’s more like looking at it as \(d^2 \propto \frac{1}{F}\). In the case of a simple inverse-linear proportionality (like angular size and distance), we would not have to worry about whether of not we need to square or square-root the change. When there is a power involved, you have to look carefully at what happens in the equation, and whether the term you are starting from is the
one with the power (example part 1, distance) which means you’ll have to square it, or if the term you
start with is not raised to a power (example part 2, Force) which means you’ll have to take the root of
the change.

**Example 3: Brightness and Distance**

An astronomer measures the light from two stars known to be very similar. Star A is 16 times dimmer
than Star B. Therefore, we expect that Star A is ___ times farther away than Star B.

1) 256
2) 16
3) 4
4) 1/256
5) 1/16
6) ¼

Equation number 20 is \[ B = \frac{L}{4\pi d^2} \]. In the question, the intrinsic power output of the stars is believed to
be the same – “known to be very similar” – so the luminosity, \( L \), is the same for both stars. This means
that the important dependence is \( B \propto \frac{1}{d^2} \). We are given the change in \( B \), so the chart looks like this:

<table>
<thead>
<tr>
<th>Star A</th>
<th>Star B</th>
<th>Thought Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16B</td>
<td>B</td>
<td>Take ( B ) as the brightness of star B. So, Star A is 1/16 ( B ).</td>
</tr>
<tr>
<td>( 16 \ d^2 )</td>
<td>( d^2 )</td>
<td>It seems weird, because in the other examples, the numbers have been in the 2(^{nd}) column, but this is right. In the question, Star B is at the “basic” distance of ( d ).</td>
</tr>
<tr>
<td>( 4 \ d )</td>
<td>( d )</td>
<td>Now, we want the change in ( d ) – so we have to take the square root for both. Star A is 4 times farther away.</td>
</tr>
</tbody>
</table>

Given values on Brightness – the non-power term – we find the change, take the square root, invert it,
and apply to distance (the squared term).

For completeness, we can look at the question in reverse:

You see light from two stars known to be very similar. From an independent method, we know that star
A is 4 times farther than Star B. Therefore, we expect that Star A is ___ times dimmer than Star B.

<table>
<thead>
<tr>
<th>Star A</th>
<th>L</th>
<th>Star B</th>
<th>Thought Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 \ d )</td>
<td>( d )</td>
<td>Start with the knowns, here in terms of the distance to B.</td>
<td></td>
</tr>
<tr>
<td>( 16 \ d^2 )</td>
<td>( d^2 )</td>
<td>Square up to apply the relation ( d^2 ) goes as ( 1/B ).</td>
<td></td>
</tr>
<tr>
<td>( 1/16 \ B )</td>
<td>( B )</td>
<td>Apply the change. Star A is 1/16 as bright; it’s 16 times dimmer.</td>
<td></td>
</tr>
</tbody>
</table>
Just to round things out, here’s the algebraic solution to these two problems:

**Version 1: ...Star A is ____ time farther away than Star B.**

First, we will need to rewrite the equation for \(d\): 

\[
\pi^4 B^4 L d = \text{and for this problem, } \frac{1}{16} B_A = B_B
\]

\[
d_A = \sqrt{\frac{L}{B_A 4\pi}} \quad \frac{L}{B_B 4\pi} = \sqrt{\frac{L}{\frac{1}{16} B_B 4\pi}} = \sqrt{\frac{1}{\frac{1}{16}}} = \frac{16}{1} = \sqrt{16} = 4
\]

The distance to star A is 4 times the distance to star B \((d_A = 4d_B)\).

**Version 2: ...Star A is ____ times dimmer than Star B.**

This time we can start from the equation as written, and that \((d_A = 4d_B)\).

\[
\frac{B_A}{B_B} = \frac{\frac{L}{4\pi d_A^2}}{\frac{L}{4\pi d_B^2}} = \frac{1}{\frac{1}{4\pi 16 d_B^2}} = \frac{1}{\frac{1}{16}} = \frac{16}{1}
\]

\[
B_A = \frac{1}{16} B_B \quad \text{Star A is 16 times dimmer than Star B.}
\]

**Final note on identifying the proportionalities in equations, and translating this to a graph.**

Consider the gravity equation. Notice that the inverse-square relationship is between \(F\) and \(d\), not \(F\) and \(d^2\). The factor applied to \(F\) is not the same as the factor applied to \(d\), instead (one over) the square root of the factor applies to \(d\). However, the factor applied to \(F\) is just 1/factor applied to \(d^2\) – those two are inverse-linear proportional. So, \(F\) and \(d\) are inverse-square proportional (more specifically, \(F\) is proportional to the inverse square of \(d\), and \(d\) is proportional to the inverse-square root of \(F\). \(F\) and \(d^2\) are just inverse (linearly) proportional. The way you group the terms in an equation is important. In the formula for volume of a sphere, \(V = 4\pi r^3\), the volume \(V\) is directly proportional to the cube of the radius. The factor of change in \(r\) must be cubed to find the factor of change in \(V\), \(V \propto r^3\). However, \(V\) and \(r^3\) are just directly proportional. Imagine substituting \(x = r^3\) into the equation for volume: \(V = 4\pi r^3 = 4\pi x\). Now it’s easy to see any change in \(V\) would be the same change in \(x\), but not in \(r\). This logic is what happens when you square \(d\) in the chart, but it is important to see how it relates back to equations. There is a practical advantage for the physics lab: two variable which are directly proportional will plot as a straight line. \(V\) vs. \(r\) is some kind of a curve; \(V\) vs. \(r^3\) is a straight line. Likewise from a’\(^3\)=b’\(^2\) (Kepler’s 3\(^{rd}\)), \(a\) vs. \(P\) is a curve, but \(a^3\) vs. \(P^2\) is a straight line. Finally, if we plotted the angular size of an object as we got close to it, \(A\) vs. \(d\) would be a curved line starting very large near the y-axis and dropping to zero at large \(d\), but \(A\) vs. \(1/d\) would increase from zero as a straight diagonal line. We’ll look a bit more at graphs later in the course.