Stellar Structure: Theory and Project

Mark Fardal
fardal@astro.umass.edu
LGRT 532A
Building Stellar Models

- We have gradually the pieces of physics together that are needed to understand stars...

- Today we take a peek at how they all fit together. This will turn out not to be simple. A star is a system where all the various aspects have to be in balance.

- Remember the nuclear timescale is much longer than the thermal or dynamical timescale, so we can assume star is static.
Equations of Stellar Structure

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho
\]

\[
\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon
\]

\[
\frac{dP(r)}{dr} = -\frac{GM(r)}{r^2} \rho
\]

\[
\frac{1}{3} \frac{d(aT^4)}{dr} = -\frac{L(r)}{4\pi r^2 c} \bar{\kappa} \rho
\]

- Also have local relations:

\[
P = P(\rho, T, Z)
\]

\[
\bar{\kappa} = \bar{\kappa}(\rho, T, Z)
\]

\[
\epsilon = \epsilon(\rho, T, Z)
\]

**IF** energy transport is radiative
Equations of Stellar Structure

Mass conservation
\[ \frac{dM(r)}{dr} = 4\pi r^2 \rho \]

Energy conservation
\[ \frac{dL(r)}{dr} = 4\pi r^2 \rho \varepsilon \]

Hydrostatic equilibrium
\[ \frac{dP(r)}{dr} = - \frac{GM(r)}{r^2} \rho \]

Energy transport
\[ \frac{1}{3} \frac{d(aT^4)}{dr} = - \frac{L(r)}{4\pi r^2} \bar{\kappa} \rho \]

- Also have local relations:
  \[ P = P(\rho, T, Z) \]
  \[ \bar{\kappa} = \bar{\kappa}(\rho, T, Z) \]
  \[ \varepsilon = \varepsilon(\rho, T, Z) \]
Equations of Stellar Structure

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho
\]

\[
\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon
\]

\[
\frac{dP(r)}{dr} = -\frac{GM(r)}{r^2} \rho = -g(r) \rho
\]

\[
\frac{dP_{rad}(r)}{dr} = -\frac{L(r) \bar{\kappa}}{4\pi r^2 c^2} \rho = -F(r) \frac{\bar{\kappa}}{c} \rho
\]

- Alternatively, use radiative pressure instead of temperature
- Note symmetry between left (mass) and right (radiation) groups
Numerical solution methods

Only practical way to get a **realistic** model of a star is with a computer solution. The main methods used are called:

- Shooting
- Fitting
- Relaxation

All of these are iterative, approximate methods.
Polytrope models

• What if we **ASSUME** $P$ scales as power-law in $\rho$? This is called a polytrope model of index $n$.

$$P = K \rho^\gamma \quad \gamma \equiv 1 + \frac{1}{n}$$

• Forget about right side equations (energy group), just look at left side (pressure group).

• This is a pair of 1st-order equations for $M(r)$, $\rho(r)$.

• Or else, a 2nd-order equation for $\rho(r)$.

$$\frac{\gamma K}{r^2} \frac{d}{dr} \left( r^2 \rho^{\gamma-2} \frac{d\rho}{dr} \right) = -4\pi G \rho$$
Polytropic Model

\[
\frac{dP}{dr} = -G \frac{M_r \rho(r)}{r^2} \quad \Rightarrow \quad \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho
\]

\[
\frac{(n+1) K_P}{4\pi G n} \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho^{(n-1)/n}} \frac{d\rho}{dr} \right) = -\rho.
\]

- Introduce two dimensionless variables:

\[
\frac{\rho}{\rho_c} = \Theta^n \quad \& \quad \xi = \frac{r}{\alpha} \quad \alpha^2 = \left[ \frac{(n+1) K_P}{4\pi G \rho_c^{(n-1)/n}} \right]
\]

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Theta}{d\xi} \right) = -\Theta^n
\]
Main sequence stars have broadly similar structures. This plot shows scaled density vs scaled radius for MS stars of mass 1.0 (red), 2.0, 4.0, 8.0.

Source: EZ-Web code (Rich Townsend)
Simple model for star

We're going to use a toy model for a main-sequence star that yields actual numbers for radius, luminosity, \( T_{\text{eff}} \), etc. This model has three main assumptions:

- Hydrogen is the fuel for the star (although the CNO cycle can help burn it). We also assume the ideal gas law.

- Stars of different masses have different radii, luminosities, etc., but all stars have the same internal shape ("homology"). For this shape we'll assume an \( n=1 \) polytrope.

- Because of this fixed shape we can't match the radiation equation throughout the star, but we can match it at one point, chosen to be the radial midpoint.
Simple model for star: pressure

- Assume $n=1$ polytrope
  \[ T \propto \rho, \quad P \propto \rho^2 \]
- This has an analytic solution
  \[
  \rho(r) = \rho_c f(x = r/r_s) \\
  T(r) = T_c f(x = r/r_s)
  \]
  \[
  f(x) = \frac{\sin(x)}{x}, \quad 0 \leq x \leq \pi
  \]
- As long as $P_c$ set correctly, hydrostatic equilibrium will be satisfied everywhere.
- Mass derivative automatically matches density too.
Simple model for star: radiation

• Use realistic energy generation rates $\epsilon(r)$.
• Energy conservation becomes simple integral $\rightarrow L(r)$.
• Radiative transfer is harder. Can only satisfy it at a single representative point. We choose $R/2$ as that point.
• At that point the gradient equations are:

$$\frac{dP(r)}{dr} = -\frac{GM(r)}{r^2}\rho$$

$$\frac{1}{3} \frac{d(\alpha T^4)}{dr} = -\frac{L(r)}{4\pi r^2} \frac{\kappa}{c}\rho$$
Simple model for star: radiation

- Use realistic energy generation rates $\epsilon(r)$.
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$$\frac{2P_c}{x_m^2 r_s} = \frac{1}{\pi x_m^3} \frac{GM \rho_c}{r_s^2}$$

$$\frac{128 \ c a T_c^4}{3\pi^5} = \frac{2}{\pi^4} \frac{L_m \kappa_m \rho_c}{r_s^2}$$
Simple model for star, ctd.

Recast model as two ratios:

\[
\frac{\text{Pressure lhs}}{\text{Pressure rhs}} = \left( \frac{2RP_c}{GM \rho_c} \right) = 1
\]

\[
\frac{\text{Radiative lhs}}{\text{Radiative rhs}} = \left( \frac{64}{3\pi} \frac{c a T_c^4 r_s}{L_m \kappa_m \rho_c} \right) = 1
\]

Or in log form:

\[
\ln \left( \frac{\text{Pressure lhs}}{\text{Pressure rhs}} \right) = \ln \left( \frac{2RP_c}{GM \rho_c} \right) = 0
\]

\[
\ln \left( \frac{\text{Radiative lhs}}{\text{Radiative rhs}} \right) = \ln \left( \frac{64}{3\pi} \frac{c a T_c^4 r_s}{L_m \kappa_m \rho_c} \right) = 0
\]

Goal is to solve these equations.
Simple model for star: trying it out

Web widgets for this simple model can be found at www.astro.umass.edu/~fardal/a335

- One version does user-specified parameters.
- The other tries to find the solution automatically.

Both versions will be used in the stellar structure project.