

Astronomy 452 Astrophysics II: Galaxies

Homework III (Due Monday, 4/13, in class)

- (i) A small ball is attached to one end of a thin massless thread whose other end goes through a small hole in the middle of a perfectly level, frictionless table. Suppose you pull the thread slowly down through the hole while the ball is revolving on the table around the hole. Prove that the work you do is the same as the energy the ball gains.
- (ii) Adiabatic contraction: Suppose a star with mass m is orbiting around a black hole of mass M in circular orbit of radius R .
 - (1) What is the orbital speed and angular momentum of the star?
 - (2) Suppose the mass of the black hole grows slowly so that its mass increases from M to $2M$. What is the angular momentum, orbital speed, and orbital radius of the stars at the end of the growth?
- (iii) The Plummer sphere is a simple, crude model for some star clusters. Its gravitational potential has the form:

$$\Phi(r) = -\frac{GM}{(r^2 + r_p^2)^{1/2}},$$

where r_p is a constant scale radius.

- (1) What is its total mass?
- (2) What is its density profile, $\rho(r)$?
- (3) When the Plummer sphere is viewed from a large distance along the z -axis, what is its surface density $\Sigma(R)$, where R is the distance from the center?
- (iv) The density profile of a singular isothermal sphere can be written as

$$\rho(r) = \rho_0(r_0/r)^2,$$

where ρ_0 and r_0 are constant.

- (1) What is the mass of the halo within a radius R from the center?
- (2) Suppose a galaxy disk (whose mass is negligible) is embedded in such a halo, what is its rotation curve, $V_{\text{rot}}(R)$?
- (v) The profile of a cold dark matter halo can usually be approximated by the Navarro-Frenk-White (NFW) profile:

$$\rho(r) = \frac{\rho_0}{(r/r_s)(1 + r/r_s)^2},$$

where ρ_0 and r_s are constant.

- (1) What is the potential of the halo at a distance R from the center, $\Phi(R)$, assuming $\Phi(R \rightarrow \infty) = 0$?
 - (2) What is the minimum velocity a star must have in order for it to escape the halo from a radius R ?
 - (3) Suppose a galaxy disk (whose mass is negligible) is embedded in such a halo, what is its rotation curve, $V_{\text{rot}}(R)$?
- (vi) For a random sample of stars in a globular cluster, the observed velocity dispersion along the line of sight is $\sigma_r = 10 \text{ km s}^{-1}$; and the surface brightness can be fit approximately by the Plummer model in Problem (iii) above with $r_p = 10 \text{ pc}$. Assuming that the cluster is spherical and contains no unseen dark matter, use the virial theorem to estimate its mass.
 - (vii) If the distribution function for stars in a spherical system depends only on their energy, i.e. if $f(\mathbf{x}, \mathbf{v}, t) = f(E)$, where $E = v^2/2 + \Phi(\mathbf{x})$. Show that the velocity dispersion of stars is isotropic, i.e. $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$.
 - (viii) The Observed Tully-Fisher relation can roughly be written as $L \propto V_{\text{max}}^4$, where L is the luminosity and V_{max} is the maximum rotational velocity of the disk. Show that this relation can be derived directly from the virial theorem if the mass to light ratio of the disk, (M/L) is proportional to $I_0^{-1/2}$ where I_0 is the central surface brightness of the disk. Hint: assuming an exponential disk.
 - (ix) To understand how spiral arms are formed in a differentially rotating disk, consider a disk of radius 1 with a flat rotation curve, i.e. with $V_{\text{rot}}(R) = \text{constant}$. Suppose the disk is in the x - y plane, with the center at the origin, and is rotating in a clockwise way. Suppose at the time $t = 0$ there is, in the disk, a density enhancement that is represented by a line along the x -axis. Please find the equation for the density enhancement at any other time $t > 0$ [Hint: use a polar coordinate system (r, ϕ) where ϕ is the angle relative to the x -axis]. Please make a sketch of this curve at the following two times: (a) when the outer disk has rotated 90° ; (b) when the outer disk has rotated 180° .
 - (x) The spider diagram and the rotation curve of a disk. In a galaxy where the potential follows that of a singular isothermal sphere, the disk rotation speed is a constant V_0 . For a disk of radius 1 with an inclination angle $i = 30^\circ$, draw a spider diagram that shows contours of constant line-of-sight velocity at 0, 0.2, 0.4, 0.6 and 0.8 times $V_0 \sin i$. How do you observe such a spider diagram for a real disk of neutral hydrogen? [Hint: an inclined disk appears as an ellipse in the sky. You can choose the long axis of the ellipse to align with the x -axis, and the short axis of the ellipse to align with the y -axis].