5.1 How about the gas component?

For stars, collisions are very rare: mean-free time is about $10^{12}\text{yr}$.

The mean-free time for gas particles in a galaxy

$$t = \frac{1}{n_{\text{gas}} \sigma v},$$

where $n$ is the number density of gas particles; $\sigma$ is the collision cross section, and $v$ is the typical velocity of gas particles (related to the temperature by $v^2 \sim kT/m$).

Inserting typical values of quantity, we have

$$t \sim 30,000\text{yr} \left( \frac{n}{10^3 \text{m}^{-3}} \right)^{-1} \left( \frac{\sigma}{10^{-19} \text{m}^2} \right)^{-1} \left( \frac{v}{10 \text{ km s}^{-1}} \right)^{-1}.$$

Thus, gas particles collide with each other very frequently.

When gas particles collide, atoms and molecules can be excited. Gas can then cool due to radiation.

Gas can then lose internal energy (or pressure) and collapse further, enhancing cooling. This is a runaway process.

The end will be a blackhole? This process can be truncated by

- Star formation
- Angular momentum of gas

Basic ideas of formation of disk and elliptical

- If star formation is not effective, gas will first settle into a thin gaseous disk: why? Cooling can get rid of energy but not angular momentum. Minimize random motion but conserve orderly motion.
- If stars form very quickly during the collapse, random-motion energy due to the collapse cannot be effectively got rid of, we have a ‘hot’ system, an elliptical.
6
Spiral galaxies

- Basic properties of disk galaxies
- Formation of galaxy disks

6.1 Exponential Disks
Except in the inner part (where bulge component may be important), the surface-brightness distribution can well be approximated by an exponential profile:

\[ I(R) = I_0 \exp \left(-\frac{R}{R_d}\right), \quad I_0 = \frac{L_d}{2\pi R_d^2}, \]

\( R \): cylindrical radius, \( R_d \): the exponential scalelength, \( I_0 \): central luminosity surface density, and \( L_d \): disk total luminosity.

Milky Way: \( R_d \approx 3.5 \text{kpc}, \text{ and } I_0 \approx 150 L_\odot \text{pc}^{-2} \) (in the V-band).

6.2 Disk-Bulge Decomposition
Many spiral galaxies contain central bulges in addition to the disk components.

Bulges have smooth light distribution and many of them have \( R^{1/4} \) light profiles. So a simple decomposition is:

\[ I(R) = I_0 \exp \left(-\frac{R}{R_d}\right) + I_{\text{eff}} \exp \left\{-7.67 \left[ \left( \frac{R}{R_{\text{eff}}} \right)^{1/4} - 1 \right] \right\}, \]

where \( R_{\text{eff}} \) is the effective (half-light) radius of the bulge.

The bulge/total ratio (in luminosity) is

\[ \frac{B}{T} = \frac{R_{\text{eff}}^2 I_{\text{eff}}}{R_{\text{eff}}^2 I_{\text{eff}} + 0.28 R_d^2 I_0}, \]

independent of distance.

6.3 The motion of stars in galaxy disk
- The mass distribution in a galaxy disk is roughly axisymmetric
- Stars have roughly circular motions
- The angular momentum of a star is conserved
Fig. 6.1. Examples of different types of galaxies. From left to right and top to bottom, NGC 4278 (E1), NGC 3377 (E6), NGC 5866 (SO), NGC 175 (SBA), NGC 6814 (Sb), NGC 4565 (Sb, edge on), NGC 5364 (Sc), Ho II (Irr I), NGC 520 (Irr II). [Photographs from the Carnegie Atlas, courtesy of A. Sandage]

Fig. 6.2. The surface brightness of galaxy disks. On large radius, the profile is roughly exponential.
6.3.1 Angular momentum
Definition:
\[ L = r \times p, \]
where \( p = mv \) is the momentum.
The change of angular momentum is due to torque:
\[
\frac{dL}{dt} = \frac{d(r \times p)}{dt} = \frac{dr}{dt} \times p + r \times \frac{dp}{dt} = \mathbf{v} \times \mathbf{p} + r \times \mathbf{F} = \mathbf{r} \times \mathbf{F}
\]
The angular momentum is conserved for stars in an axisymmetric potential.

6.3.2 Equation of motion in a disk
Assuming axisymmetry around the \( z \)-axis.
Conservation of angular momentum:
\[ L_z = mRv = mR^2 \dot{\phi} = \text{const} \]
Differentiate once with respect to time gives
\[ 2R\ddot{\phi} + R^2 \dot{\phi} = 0 \]
\[ 2\ddot{R} + \dot{\phi}^2 = -R\dot{\phi} \]
The disk potential is \( \Phi = \Phi(R, z) \)
Motion of a star is therefore
\[ \mathbf{F} = m\mathbf{a} = m\ddot{R}. \]
We can write
\[ \mathbf{R} = x + y = R \cos \phi \mathbf{e}_x + R \sin \phi \mathbf{e}_y \]
\[ \dot{\mathbf{R}} = (\dot{R} \cos \phi - R \sin \phi \dot{\phi})\mathbf{e}_x + (\dot{R} \sin \phi + R \cos \phi \dot{\phi})\mathbf{e}_y \]
Thus
\[ \ddot{\mathbf{R}} = \ddot{R} \mathbf{e}_R - R \dot{\phi}^2 \mathbf{e}_R - (2\ddot{\phi} \sin \phi + R \sin \phi \dot{\phi})\mathbf{e}_x + (2\ddot{\phi} \cos \phi + R \cos \phi \dot{\phi})\mathbf{e}_y = \ddot{R} \mathbf{e}_R - R \dot{\phi}^2 \mathbf{e}_R - \sin \phi (2\ddot{\phi} + R \ddot{\phi})\mathbf{e}_x + \cos \phi (2\ddot{\phi} + R \ddot{\phi})\mathbf{e}_y \]
Using conservation of angular momentum, we finally get
\[ \ddot{\mathbf{R}} = (\ddot{R} - R \dot{\phi}^2)\mathbf{e}_R \]
In the above \( \mathbf{e}_R = \mathbf{e}_x \cos \phi + \mathbf{e}_y \sin \phi \) is the unit vector in the radial direction.
Thus:
\[ F_R = m(\ddot{R} - R \dot{\phi}^2) \]
Or
\[ m\ddot{R} = F_R + mR\dot{\phi}^2, \]
where \( F_R \) is the gravitational force, and \( mR\dot{\phi}^2 \) is the ‘centrifugal force’ (not a real force).

Since
\[ F_R = -m \frac{\partial \Phi}{\partial R} \]
and
\[ \frac{\partial \Phi}{\partial R} = -\frac{L_z^2}{m^2 R^3}, \]
we have
\[ \ddot{R} = R\dot{\phi}^2 - \frac{\partial \Phi}{\partial R} \equiv \frac{\partial \Phi_{\text{eff}}}{\partial R} \]
where
\[ \Phi_{\text{eff}} \equiv \Phi + \frac{L_z^2}{2R^2}, \]
with \( L_z = L_z/m \) being the specific angular momentum.

Energy conservation:
\[ -\frac{\partial \Phi_{\text{eff}}}{\partial R} = \ddot{R} = \frac{d}{dt} \frac{dR}{dt} = \frac{dR}{dt} \frac{d}{dt} \frac{dR}{dt} = \frac{1}{2} \frac{d}{dt} \left( \frac{dR}{dt} \right)^2 \]
This gives

\[ \frac{1}{2} \ddot{R}^2 + \Phi_{\text{eff}} = \mathcal{E} = \text{const.} \]

\( \mathcal{E} < 0 \) bound orbit
\( \mathcal{E} > 0 \) un-bound orbit
Note that \( \dot{R}^2/2 \geq 0 \) and so \( \mathcal{E} \geq -\Phi_{\text{eff}} \).
The angular momentum barrier at \( \mathcal{E} \geq -\Phi_{\text{eff}} \), which corresponds to a
perigalactic radius (the minimum radius the star can reach)
The equilibrium radius \( R_g \) at which \( \dot{R} = 0 \), i.e. \( \Phi_{\text{eff}} = \text{constant} \), and the
force is also zero, which means

\[ \frac{\partial \Phi_{\text{eff}}}{\partial R} = 0 \]

Thus

\[ \left( \frac{\partial \Phi}{\partial R} \right)_{(R_g,z=0)} = \frac{L_z^2}{R_g^2} = R_g \Omega^2(R_g), \]

where \( \Omega(R_g) = \dot{\phi}(R_g) \) is the angular speed
Example for a point mass potential

\[ \frac{GM}{R_g^2} = R_g \Omega^2(R_g) \]

i.e. gravitational force balance centrifugal force.

### 6.3.3 Epicycle motion

Suppose there is deviation from \( R \) from \( R_g \), how does the star move then?
Write \( R = R_g + r \) and assume \( r \ll R_g \).

From

\[ \ddot{R} = -\frac{\partial \Phi_{\text{eff}}(R)}{\partial R} \]

we get

\[ \ddot{r} = -\frac{\partial \Phi_{\text{eff}}(R_g + r)}{\partial R} \approx -\frac{\partial \Phi_{\text{eff}}(R_g)}{\partial R} - \left[ \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right]_{R_g} r. \]

Thus,

\[ \ddot{r} = -\kappa^2(R_g) r, \]

where

\[ \kappa^2(R_g) = \left[ \frac{\partial^2 \Phi}{\partial R^2} \right]_{R_g} = \left[ \frac{\partial^2 \Phi}{\partial R^2} \right]_{R_g} + \frac{3L_z^2}{R_g^4}. \]

The above equation has the solution

\[ r = X \cos(\kappa t + \psi). \]

If \( \kappa^2 > 0 \) then it is harmonic oscillation, with \( \kappa \) being the epicycle frequency;
If \( \kappa^2 < 0 \) then it is an unstable solution.
Fig. 6.4. $\Phi_{\text{eff}}$ as a function of $R$ in two cases, one with $\kappa^2 = (\partial^2 \Phi_{\text{eff}} / \partial R^2)_{R_0} > 0$, and another with $\kappa^2 < 0$.

Write the ratio $\kappa / \Omega = m$. If $m$ is an integer, then the orbit closes after $m$ epicycle.

Pattern speed $\Omega_p$ (of bar or spiral arms).

In the frame co-rotate with the pattern, we can define

$$\kappa \Omega - \Omega_p = m,$$

so that

$$\Omega_p = \Omega + \kappa / m.$$

- Co-rotation: $m \to \infty$, i.e. $\Omega_p = \Omega$. In this case stars co-rotate with the pattern;
- Inner Lindblad resonance, $m = -2$: $\Omega_p = \Omega - \kappa/2$. Since $\Omega > \Omega_p$, stars overtake potential (due to bar or spiral arms);
- Outer Lindblad resonance, $m = 2$: $\Omega_p = \Omega + \kappa/2$. Since $\Omega < \Omega_p$, stars lag behind potential.
6.4 Disk Vertical Structure

Disks are not infinitesimally thin but have vertical ($z$-) structure. The luminosity density as a function of $z$ and $R$ can fit reasonably well by

$$\rho(R, z) = \rho_0(R, 0) \text{sech}^2 \left( \frac{z}{2H_d} \right),$$

where $H_d$ is called the scale height (assumed to be independent of $R$).

For our own Galaxy, the scaleheight of the young (thin) disk is $\sim 0.3$ kpc, while that for the old (thick) disk can be as large as $\sim 1.3$ kpc.

What determines the disk thickness: a balance between gravity and velocity dispersion.

A simple model: *locally* isothermal sheet. The distribution function of stars in the vertical $z$-direction is Maxwellian,

$$f = \frac{1}{\sqrt{2\pi}\sigma_z} \exp \left( -\frac{E_z}{\sigma_z^2} \right),$$

where $\sigma_z$ is the velocity dispersion of stars in the vertical direction, $E_z \equiv v_z^2/2 + \Phi(R, z)$.

For a very flattened system, Poisson’s equation,

$$\partial^2\Phi/\partial z^2 = 4\pi G \rho(R, z)$$

For the given, it can be written as

$$\frac{d^2\phi}{d\zeta^2} = \frac{1}{2} e^{-\phi},$$

where

$$\phi \equiv \frac{\Phi}{\sigma_z^2}, \quad \zeta \equiv \frac{z}{z_0}, \quad z_0 \equiv \frac{\sigma_z}{\sqrt{8\pi G \rho(R, 0)}}.$$

The boundary conditions: both $\phi$ and its derivative $d\phi/d\zeta$ are zero at $z = 0$.

The solution

$$\rho(R, z) = \rho(R, 0) \text{sech}^2 \left( z/H_d \right),$$

where $H_d \equiv 2z_0$ is the scale-height of the disk.

Surface density and mid-plane density:

$$\rho(R, 0) = \frac{\Sigma(R)}{2H_d},$$

Thus,

$$H_d = \frac{\sigma_z^2}{\pi G \Sigma(R)}.$$

6.5 Rotation curves and dark halos

Stars and cold gas in galaxy disks are rotating around the axes of the disks.

The rotation is described by its rotation curve $[V_{\text{rot}}(R)]$.

Disk rotation curves can be measured (by long-slit spectroscopies) either at the optical wavelengths from emission lines (e.g. Hα) in HII regions, or at radio wavelengths from the 21-cm emission line from neutral (HI) gas.
Fig. 6.5. The rotation curve of NGC3198.

The rotation curve is a direct measure of the gravitational force (and mass). Assuming spherical symmetry, the total mass within $r$ is

$$M(r) = rV_{\text{rot}}^2(r)/G.$$  \hfill (6.1)

In the outer region where $V_{\text{rot}}(r)$ is roughly a constant, $M(r) \propto r$.

Thus, the total mass depends significantly on how far the flat rotation curve extends.

### 6.6 The formation of Spiral Arms

Spiral galaxies show variety of spiral structures: Some are grand-design arms (usually two); others are arm segments.

- Small material arms: due to shear of local perturbation by differential rotation
- Grand-designed arms: spiral density waves or excited by nearby satellites.

Differential rotation is the key.

### 6.7 Disk Angular Momentum

Disks are supported by rotation. The specific angular momentum of disk material at $R$ is

$$\mathcal{J}_d(R) = V_c(R)R.$$ 

where $V_c(R)$ is the rotation curve.

In the outer part where the rotation curve is flat, $\mathcal{J}_d(R) \propto R$; while for the part where $V_c(R) \propto R$, $\mathcal{J}_d(R) \propto R^2$.

The total angular momentum of the disk is

$$J_d = 2\pi \int_0^\infty V_c(R)\Sigma(R)R^2dR.$$ 

Assuming that disk material within $R$ has specific angular momentum less than $\mathcal{J}_d(R)$, then

$$M_d(<\mathcal{J}_d) = M_d(R).$$
Fig. 6.6. Spiral arm produced by differential rotation.

Fig. 6.7. Spiral pattern in density wave. Consider an undisturbed stellar disk where all stars are in circular orbits with angular velocity $\Omega(R)$. If the disk is subjected to a global perturbation corresponds to a deformation of the circular orbits into elliptical orbits. In general, the perturbation changes with time, and so the major axis of an elliptical orbit will rotate with time, with some angular speed $\Omega_p$. If the rotations of all the orbits have the same phase in $\varphi$, then we have an aligned set of elliptical orbits rotating with a pattern speed $\Omega_p$. If, however, the phase in $\varphi$ changes smoothly with $R$, a two-armed structure, which rotates at the pattern speed, will be produced.
For a flat rotation curve \( V_c(R) = V_c \),

\[
M_d(< J_d) = M_d \left[ 1 - \left( 1 + \frac{2M_d J_d}{J_d} \right) \exp \left( -\frac{2M_d J_d}{J_d} \right) \right],
\]

where \( J_d = 2M_d R_d V_c \) is the total angular momentum of an exponential disk with flat rotation curve.

6.8 The Formation of Galaxy Disks
A natural way to form a disk is through dissipational collapse of gas cloud with some initial angular momentum in an extended dark matter halo.

Effective cooling of a gas cloud causes it to approach a state with energy as low as possible while conserving angular momentum.

The preferred state is a rotating thin disk.

6.8.1 What determines the disk sizes

\[
J_d(R) = V_c(R) R.
\]

So

\[
R_d \propto J/V_c.
\]

Disk size in a halo is determined by the specific angular momentum of the gas.

Where does the angular momentum come from? Tidal field of the mass distribution in the universe.

6.9 The Tully–Fisher Relation

6.9.1 Observations
Disk galaxies are observed to cover a large range of \( L, R_d, V_c \) and rotation-curve shape, why do they obey a tight Tully-Fisher relation. In the I-band

\[
M_I - 5 \log h = -21.00 - 7.68 (\log W - 2.5).
\]

For a fixed \( V_{\text{max}} \approx W/2 \) the scatter in \( L \) is about 50%.

The tight TF relation implies a close relation between the total gravitational mass and the total amount of stars that can form.

Why?

6.9.2 Model I: self–gravitating disk, no dark matter
Consider a self-gravitating exponential disk, \( R \approx 2.16R_d, V^2_{\text{max}} \approx 2.5G\Sigma_0 R_d \)

Assuming a disk mass-to-light ratio \( \Upsilon \equiv L_d/M_d \), we have

\[
L_d \approx B \left( \frac{V_{\text{max}}}{200 \text{ km s}^{-1}} \right)^4 \left( \frac{I_0}{100 \text{ L}_\odot \text{ pc}^{-2}} \right)^{-1} \left( \frac{\Upsilon_d}{\Upsilon\odot h} \right)^{-2},
\]

\[ I = \Sigma \Upsilon_d, \quad R = 8.5 \times 10^{11} h^{-2} L_\odot \]

This looks quite like the observed TF relation. There are two problems:
Fig. 6.8. The Tully-Fisher relation in the $I$-band. $W$ is the linewidth of the HI 21 cm line and is about twice $V_{\text{max}}$.

[Adopted from Giovanelli et al.]

(1) Disk is too massive for a given $V_{\text{max}}$;  
(2) The expected scatter in $L_d$ for given $V_{\text{max}}$ is too big, because it is the same as that in $I_0$ ($\sim 1$ mag)

6.9.3 Model II: Disks in Massive Dark Haloes

As another extreme, we may assume disk gravity to be negligible, and disk rotation curves are determined entirely by dark haloes. This can happen if dark haloes are very massive and concentrated.

The Tully-Fisher relation expected from this model can be obtained from

$$L_d = A \left( \frac{V_c}{250 \text{ km s}^{-1}} \right)^{\alpha}, \quad \alpha = 3, \quad A = 1.7 \times 10^{11} h^{-1} L_\odot \left[ \frac{r_d^{-1} m_d}{0.05} \right] \left[ \frac{H(z)}{H_0} \right]^{-1}$$

This is quite similar to the observed Tully-Fisher relation, provided all the factors in the brackets are of order unity.

Problem: requires a constant $m_d \sim 0.05 < \Omega_{B,0}/\Omega_0$! Not natural.
6.9.4 Disk/halo interaction

6.10 Disk Instability and Star Formation

If the disk is too heavy, it will be unstable (cannot be supported by pressure). An unstable disk will fragment, forming dense gas clouds and stars.

Toomre's $Q$ parameter:

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma}.$$ 

Disk should be locally unstable to form stars

$$Q < 1, \quad \Sigma > \Sigma_{\text{crit}} = \frac{c_s \kappa}{\pi G}$$

Empirical relation:

$$\Sigma_{\text{SFR}} \propto \Sigma_{\text{cold}}^{1.4}.$$