Interferometry: I. Basic Concepts

Incident Field:

The incident radiation from the sky is a function of the direction (ξ) and of time (t). The electric field of this radiation $E(\xi, t)$ can be expressed in terms of the spatial frequency spectrum $e(u, t)$ by

$$E(\xi, t) = \int_{-\infty}^{+\infty} e(u, t) e^{i2\pi\xi u} du$$

The inverse FT with respect to ξ and u gives

$$e(u, t) = \int_{-\infty}^{+\infty} E(\xi, t) e^{-i2\pi\xi u} d\xi$$

Source Coherence Function:

$$\gamma(\xi_1, \xi_2, \tau) = \lim_{T\to\infty} \frac{1}{2T} \int E_T(\xi_1, t) E_T^*(\xi_2, t-\tau) dt$$
$$= < E_T(\xi_1, t) E_T^*(\xi_2, t-\tau) >$$

where $E(\xi, t) =$ E field at source position ξ.

"cross correlation of signals received from two different points on the source at two different times."

- If $\xi_1 = \xi_2 = \xi$,
  $$\gamma(\xi, \tau) = \lim_{T\to\infty} \frac{1}{2T} \int E_T(\xi, t) E_T^*(\xi, t-\tau) dt$$
  ["auto-correlation function towards ξ"]

- If $\tau = 0$, $\gamma(\xi, 0) = \lim_{T\to\infty} \frac{1}{2T} \int |E_T(\xi, t)|^2 dt$
  ["average brightness towards ξ"]

Normalized source coherence function:

$$\gamma_N(\xi_1, \xi_2, \tau) = \frac{\gamma(\xi_1, \xi_2, \tau)}{\sqrt{\gamma(\xi_1, 0)\gamma(\xi_2, 0)}}$$

Spatial Coherence Function:

$$\Gamma(u_1, u_2, \tau) =_{T\to\infty}^{lim} \frac{1}{2T} \int e_T(u_1, t) e_T^*(u_2, t-\tau) dt$$
$$=< e_T(u_1, t) e_T^*(u_2, t-\tau) >$$

$u \equiv$ spatial coordinate in λ, normal to $\xi = 0$

$$e(u, t) = \int_{-\infty}^{+\infty} E(\xi, t) e^{-i2\pi u\xi} d\xi$$

Because of this F.T. relation,

$$\Gamma(u_1, u_2, \tau) = \int \int \gamma(\xi_1, \xi_2, \tau) e^{-i2\pi(u_1\xi_1 - u_2\xi_2)} d\xi_1 d\xi_2$$

Completely Incoherent Source:

$$< E_T(\xi_1) E_T^*(\xi_2) >= 0 \ for \ \xi_1 \neq \xi_2$$

$$\gamma(\xi_1, \xi_2, \tau) = \gamma(\xi_1, \tau) \delta(\xi_1 - \xi_2)$$
$$\Gamma(u, \tau) = \int_{-\infty}^{+\infty} \gamma(\xi, \tau) e^{-i2\pi u\xi} d\xi$$

where $u = u_1 - u_2$ and $\xi = \xi_1 - \xi_2$

At $\tau = 0$, $\gamma(\xi, 0) =< |E_T(\xi)|^2 >$ and

$$\boxed{\Gamma(u, 0) = \int_{-\infty}^{+\infty} < |E_T(\xi)|^2 > e^{-i2\pi u\xi} d\xi}$$

"van Cittert-Zernike Theorem"

Inverting the Measurement Equation

$\vec{S_0}$ = phase reference position = (0, 0, 1)
$\vec{S}$ = source position = $(\xi,\ \eta,\ \sqrt{1-\xi^2-\eta^2})$
$\vec{B_\lambda}$ = (u, v, w) = baseline vector
$\vec{\sigma} = \vec{S} - \vec{S_0} = (\xi,\ \eta,\ \sqrt{1-\xi^2-\eta^2} - 1)$

"visibility": $V(u,v,w) \equiv \Gamma(u, \tau = 0)$
$$= \int\int I(\xi,\eta) e^{-i2\pi(\vec{B_\lambda}\cdot\vec{\sigma})} d\Omega$$

$$\vec{B_\lambda}\cdot\vec{\sigma} = (u\xi + v\eta + w\sqrt{1-\xi^2-\eta^2} - 1)$$
$$d\Omega = \frac{d\xi d\eta}{\sqrt{1-\xi^2-\eta^2}}$$

$$V(u,v,w) = \int\int \frac{I(\xi,\eta)}{\sqrt{1-\xi^2-\eta^2}} e^{-i2\pi(u\xi+v\eta+w[\sqrt{1-\xi^2-\eta^2}-1])} d\xi d\eta$$

This is **NOT** a Fourier Transform!!

Limiting Case:

If source extent is limited to a small area around $\vec{S_0}$ so that $\vec{S} \sim \vec{S_0}$,

$$\sqrt{1-\xi^2-\eta^2} - 1 \simeq -\frac{1}{2}(\xi^2+\eta^2) \sim 0$$

$$V(u,v,w) = \int\int \frac{I(\xi,\eta)}{\sqrt{1-\xi^2-\eta^2}} e^{-i2\pi(u\xi+v\eta)} d\xi d\eta$$
$$= V(u,v,0)$$

$$\boxed{V(u,v) = \int\int \frac{I(\xi,\eta)}{\sqrt{1-\xi^2-\eta^2}} e^{-i2\pi(u\xi+v\eta)} d\xi d\eta}$$

$\Rightarrow$ This **IS** a 2-D Fourier Transform!

Note:

$I(\xi,\eta)$ is real. $V(u,v)$ is complex.

$$V(u,v) = \int I(\vec{\sigma}) e^{-i2\pi(\vec{B_\lambda}\cdot\vec{\sigma})} d\Omega$$
$$= \int I(\vec{\sigma}) cos(2\pi \vec{B_\lambda}\cdot\vec{\sigma}) d\Omega$$
$$+ i \int I(\vec{\sigma}) sin(2\pi \vec{B_\lambda}\cdot\vec{\sigma}) d\Omega$$

Measure $V(u,v)$ $\Rightarrow$ Recover $I(\vec{\sigma})$ using F.T.!

II. Practical Issues

<u>RF Interferometer:</u>

cross-correlation output from the correlator:

$$r(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} V_1(t) V_2(t-\tau) dt$$

In general, for $\tau_g = \vec{B} \cdot \vec{S_0}/c$,

$$r(\tau_g) = A_0 \Delta\nu |V| cos(2\pi \vec{B_\lambda} \cdot \vec{S_0} - \phi_s)$$

- amplitude $|V|$ relates to source strength
- phase ϕ_s relates to source location wrto $\vec{S_0}$
- antenna area A_0 and bandwidth $\Delta\nu$ are instrumental

Wiener-Khinchin relation:

$$\int_{-\infty}^{+\infty} r(\tau) e^{-i2\pi\nu\tau} d\tau = |H(\nu)|^2$$

$$and \int_{-\infty}^{+\infty} r(\tau) |H(\nu)|^2 e^{i2\pi\nu\tau} d\tau = r(\tau)$$

where $H(\nu)$ is amplitude (voltage) response and $|H(\nu)|^2$ is power spectrum (**instrumental bandpass**).

Finite Bandwidth:

Real interferometers have finite bandwidths, and the correlator output is an integral over frequency.

$$V = \int I_\nu [\frac{1}{\Delta\nu} \int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2} e^{-i2\pi\nu\tau_g} d\nu] d\Omega$$

$$= \int I_\nu e^{-i2\pi\nu_0\tau_g} \frac{sin(\pi\tau_g\Delta\nu)}{\pi\tau_g\Delta\nu} d\Omega$$

To avoid loss of amplitude (sensitivity), we must track delay.

Delay Tracking (“fringe stopping”):

- reduce “bandwidth smearing”
- 2-D FT requires $\tau = 0$
  - remove instrumental delay offset τ_i
  - remove geometrical delay τ_g

FOV and Resolution:

- Field of View $= \lambda/D$

Example: $\nu = 230$ GHz, $D = 6$ m
$FOV = \frac{3\times10^{10}}{(230\times10^{9})(600)} = 2.2 \times 10^{-4} = 45''$

- Resolution: $\theta_{syn} \sim \lambda/B_{max}$

Example: $\nu = 230$ GHz, $B_{max} = 300$ m
$\theta_{syn} = \frac{3\times10^{10}}{(230\times10^{9})(30000)} = 4.3 \times 10^{-6} = 0.9''$

Sensitivity:

$$\sigma = \frac{2kT_{sys}}{\eta_q A\sqrt{N_A(N_A-1)\tau\Delta\nu}}$$

Example: $T_{sys} = 200$ K, $N_A = 8$, $A = \frac{\pi}{4}(600)^2$, $\Delta\nu = 4$ GHz, $\eta_q = 0.88$, $\tau = 60$ seconds

$$\Rightarrow \sigma = 6 \text{ mJy in 1 minute}$$

Baseline and uv-Coverage:

Standard sampling requirement applies: to "map" a source, sample the uv-space as fully as possible.

uv-sampling function $\rightleftharpoons$ PSF

(i.e., poor uv-sampling limits deconvolution.)

- astronomical coordinate system fixed on source
- a fixed baseline rotates with Earth, and so does the projected baseline
  $\Rightarrow$ a fixed baseline tracks an arc in uv-space
- since $I(l, m)$ is real, $V(-u, -v) = V^*(u, v)$
  $\Rightarrow$ actually tracks two arcs in uv-space