

III. Imaging and Deconvolution

Fourier Transform Imaging

A Fourier Transform relation exists between the sky brightness I , the primary beam pattern A , and visibility V :

$$A(l, m)I(l, m) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} V(u, v) e^{2\pi i(ul+vm)} dudv$$

“Dirty Image”:

$$I^D(l, m) \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(u, v) V'(u, v) e^{2\pi i(ul+vm)} dudv$$

where $S(u, v)$ denotes the sampling function and $V'(u, v)$ the observed (noisy) visibility. In a brute force evaluation, a DFT can be made at every $N \times N$ grid as

$$I^D(l, m) = \frac{1}{M} \sum_{k=1}^M V'(u_k, v_k) e^{2\pi i(u_k l + v_k m)}.$$

Sampling Function:

$$S(u, v) = \sum_{k=1}^M \delta(u - u_k, v - v_k)$$

Sampled Visibility Function:

$$V^S(u, v) \equiv \sum_{k=1}^M \delta(u - u_k, v - v_k) V'(u_k, v_k)$$

$$\text{i.e., } V^S = SV'$$

Let \mathbf{F} denote the F.T. operator:

$$I^D = \mathbf{F}V^S = \mathbf{F}(SV')$$

By the *convolution theorem*:

$$I^D = \mathbf{F}S * \mathbf{F}V'$$

For a point source of unit strength at (l_0, m_0) ,

$$|V'(u, v)| = 1$$

$$\mathbf{F}V'(l, m) = \delta(l - l_0, m - m_0)$$

$$\Rightarrow \mathbf{F}S * \mathbf{F}V' = \mathbf{F}S * \delta = \mathbf{F}S$$

Thus, synthesized beam $B = \mathbf{F}S$.
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Weighted Sampling Function:

$$W(u, v) = \sum_{k=1}^M R_k T_k D_k \delta(u - u_k, v - v_k)$$

- R_k = reliability weight (e.g. $\frac{\tau \Delta \nu}{T_{sys}^2}$)
- T_k = tapering weight (beam shaping)
- D_k = density weight
 - “natural weighting”: $D_k = 1$
 - “uniform weighting”: $D_k = \frac{1}{N_s(k)}$

Weighted, Sampled Visibility Function:

$$V^W(u, v) \equiv \sum_{k=1}^M R_k T_k D_k \delta(u - u_k, v - v_k) V'(u_k, v_k)$$

$$i.e., V^W = W V'$$

In this case, synthesized beam is

$$B = \mathbf{F}W$$

Deconvolution

$$V(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(l, m) e^{-2\pi i(ul+vm)} dl dm$$

In general $I^D = B * \mathbf{F}V$, and if I^D and B are known, it is mathematically possible to recover $I \equiv \mathbf{F}V$ by Fourier Transferring I^D , dividing by $\mathbf{F}^{-1}B$, and Fourier Transform back, i.e.

$$I \equiv \mathbf{F}V = \mathbf{F}^{-1} \left(\frac{\mathbf{F}I^D}{\mathbf{F}^{-1}B} \right)$$

In practice, uv-sampling is finite and $\mathbf{F}^{-1}B = W$ is zero at many places, and the division by W is not possible. Therefore the recovery of $I(l, m)$ is not possible through a linear transform.

Instead, a non-linear deconvolution might be done by constructing a 2-D model image \hat{I} that satisfies,

$$\hat{V}(u, v) = \sum_{p=1}^{N_l} \sum_{q=1}^{N_m} \hat{I}(p\Delta l, q\Delta m) e^{-2\pi i(pu\Delta l + qu\Delta m)}$$

which is related to true visibility as

$$V(u, v) = W(u, v)(\hat{V}(u, v) + \epsilon(u, v))$$

where $\epsilon(u, v)$ is a complex, random error.

By the convolution theorem,

$$I_{p,q}^D = \sum_{p',q'} B_{p-p',q-q'} \hat{I}_{p'q'} + E_{p'q'}$$

Non-Uniqueness:

Some of the spatial frequencies allowed in the model are not present in the data, and changing the amplitude of the corresponding sinusoids in I have no effect on the fit to the data. In particular, any function Z that satisfies $B * Z = 0$ can be added to the solution and satisfy the convolution equation.

The CLEAN Algorithm

Approximate an image as an ensemble of point sources in an otherwise blank field and deconvolve each point source using a simple iterative approach. The final deconvolved image (the CLEAN image) is the sum of these CLEAN components convolved with a (Gaussian) CLEAN beam.

- Högbom (1974) algorithm
 1. find the peak in the dirty image
 2. subtract $\gamma I_{p,q}^D * B$ at the peak position ($\gamma \leq 1$)
 3. construct \hat{I} using the position and magnitude of the subtracted component
 4. loop back until some threshold level is reached
 5. convolve \hat{I} with an idealized CLEAN beam
 6. add the residual noise in the dirty map to \hat{I} .

- Clark (1980) algorithm – the time consuming step of shifting and scaling of dirty beam is done as a convolution using a 2-D FFT.
- Cotton-Schwab algorithm (Schwab 1984) – similar to Clark algorithm but CLEAN components are subtracted on the ungridded visibility data (reduced aliasing noise and gridding errors).
- Major shortcomings:
 - highly non-linear process
 - extended sources poorly modeled
 - “noise” analysis difficult

Maximum Entropy Method (MEM)

“Entropy” H is defined as

$$H = - \sum_k I_k \ln \frac{I_k}{M_k}$$

where M_k is a “default” image incorporating any *a priori* knowledge. Maximize H subject to the constraint that

$$\chi^2 = \sum_k \frac{|V(u_k, v_k) - \hat{V}(u_k, v_k)|^2}{\sigma_{V(u_k, v_k)}^2}$$

be equal to its expected value.

Major shortcomings:

- image tends to be smooth
- varying resolution with S/N
- bias and positivity