Lecture 3: Digital Signal Processing

Probability Density $p(x)$:

probability that at any moment of time the value of the process $x(t)$ falls within an interval $[x - \frac{1}{2}dx,\ x + \frac{1}{2}dx]$.

Expected Value $E\{x\}$:

$$E\{x\} = \int_{-\infty}^{+\infty} xp(x)dx$$

$$E\{f(x)\} = \int_{-\infty}^{+\infty} f(x)p(x)dx$$

Example: mean $\mu = E\{x\}$

dispersion $\sigma^2 = E\{x^2\} - E^2\{x\}$

Time Average $< f(x) >$:

$$< f(x) > = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f(x(t))dt$$

$$< f(x) >_T = \frac{1}{T} \int_{-T/2}^{+T/2} f(x(t))dt$$

Fourier Transform:

$$\text{if } f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\nu) e^{i2\pi\nu t} d\nu,$$

$$\text{then } F(\nu) = \int_{-\infty}^{+\infty} f(t) e^{-i2\pi\nu t} dt$$

$$X_T(\nu) = \int_{-T/2}^{+T/2} x(t) e^{-i2\pi\nu t} dt$$

<u>Example</u>: F.T. of a Finite Pulse

$$f(t) = 1 \text{ for } |t| < T_0$$
$$= 0 \text{ for } |t| \geq T_0$$

$$F(\omega) = \int_{-T_0}^{+T_0} e^{i\omega t} dt = \frac{1}{-i\omega} e^{i\omega t}|_{-T_0}^{+T_0}$$
$$= 2T_0 \frac{sin\ \omega T_0}{\omega T_0}$$

Auto-correlation Function (ACF):

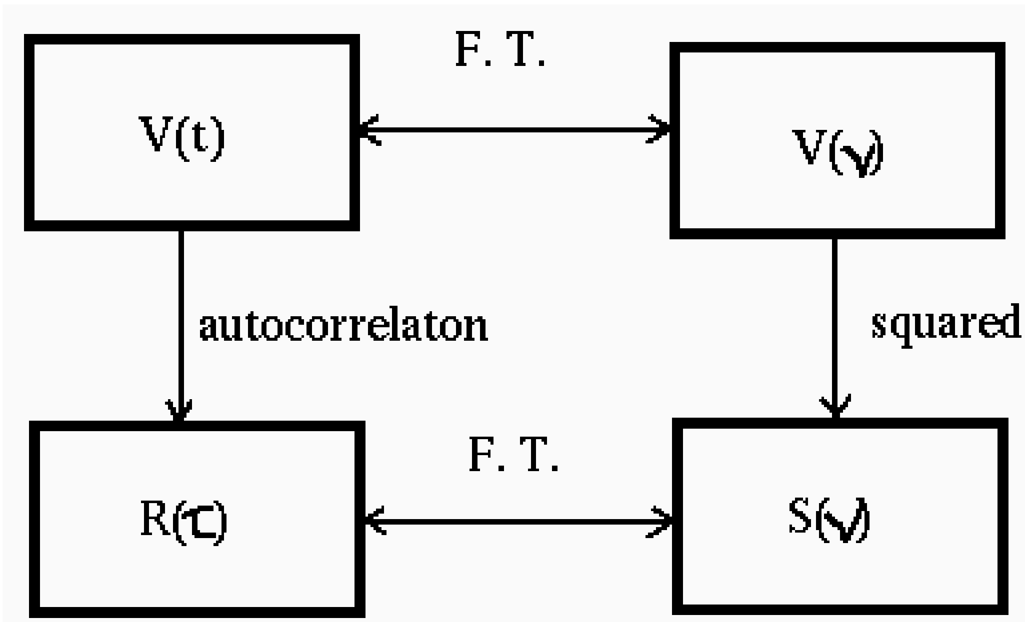
$$R_T(\tau) = E_T\{x(s)x(s+\tau)\}$$
$$= \int_{-T/2}^{+T/2} x(s)x(s+\tau)ds$$

Example: ACF of a Finite Pulse

$$f(t) = 1 \text{ for } |t| < T_0$$
$$= 0 \text{ for } |t| \geq T_0$$

$$R(\tau) = \int_{-T_0}^{+T_0} f(t)f(t+\tau)dt$$

Wiener-Khinchin Theorem



$$V_T(\nu) = \int_{-T/2}^{+T/2} V(t) e^{-i2\pi\nu t} dt$$

$$S(\nu) =_{T\to\infty}^{lim} \frac{1}{T} E_T\{V^2(\nu)\}$$

$$E_T\{V^2(\nu)\} = E\{\int\int_{-T/2}^{+T/2} V(s)V(t) e^{-i2\pi\nu(t-s)} ds dt\}$$

$$\text{since } R_T(\tau) = \int_{-T/2}^{+T/2} V(t)V(t-\tau) dt$$

$$S(\nu) = \int_{-\infty}^{+\infty} R(\tau) e^{-i2\pi\nu\tau} d\tau$$

$$\text{and } R(\tau) = \int_{-\infty}^{+\infty} S(\nu) e^{i2\pi\nu\tau} d\nu$$

Digital Spectrometers

- $V(t)$ and $V(\nu)$ are Fourier pairs (Fourier transform spectrometer, FTS)
- $R(\tau)$ and $S(\nu)$ are Fourier pairs (W-K theorem – autocorrelation spectrometer)
- Nyquist's sampling: $\Delta t \leq \frac{1}{2B}$
- digital encoding of $V(t)$ for FFT means B restricted by sampling speed
- finite sampling length limitations

2-Level Autocorrelation Spectrometer

$$y(t) = +1 \text{ for } x(t) \geq 0$$
$$= -1 \text{ for } x(t) < 0$$

$$R_y(\tau) = E\{y(t+\tau)y(t)\}$$

For a Gaussian process, this can be derived as

$$R_y(\tau) = \frac{2}{\pi} arcsin \frac{R_x(\tau)}{R_x(0)}$$

and, $R_x(\tau) = R_x(0) sin[\pi/2 \ R_y(\tau)]$

Finite Sampling Length:

$\rightarrow$ filter or apply weighting

$$e.g.\ w(\tau) = 1 \text{ for } |\tau| \leq \tau_m 0$$

$$= 0 \quad \text{otherwise}$$

$$\boxed{S = S(\nu) \otimes W(\nu)}$$

$W(\nu)$ determines spectral resolution,

$$\Delta\nu = \frac{0.605}{\tau_m}$$

Higher Bit Sampling

number of bits sampling	Efficiency at 1/2B	at 1/4B
1	0.64	0.74
2	0.81	0.89
3	0.88	0.94
∞	1.00	1.00