1 Stars: key concepts

This following summarizes important points that I covered in class or are explained in your textbook. Your book will give you additional information about any of these concepts. Use the list a guide to studying. I realize that this is a long list but I've tried to make it comprehensive. Exam #2 will only be able to cover some fraction of these, of course.

1. Understand how the Sun produces energy
2. Know what a stellar model is and be able to explain the relationship between pressure, energy generation, energy transport and energy generation
3. Understand how astronomers use the parallaxes of stars to measure their distances
4. Understand how to determine a star’s luminosity from its brightness and distance
5. Know the difference between apparent magnitude and absolute magnitude
6. Be able to explain the relationship between a star’s color and its surface temperature
7. Understand how astronomers use the spectra of stars to reveal their chemical compositions and surface temperatures
8. Know the sequence of spectral classes of stars
9. Know the relationship among a star’s luminosity, radius, and surface temperature
10. Know what a Hertzsprung-Russell (H-R) diagram is and the main groupings of stars that appear on it
11. Understand how the details of a star’s spectrum reveal what kind of star it is, its luminosity, and its distance
12. Be able to describe visual binary stars and explain how they provide information about stellar masses
13. Know the mass-luminosity relation for main-sequence stars binaries, and eclipsing binaries are
14. Understand why stars have a definite life span

15. Know what the interstellar medium is and what kind of matter constitutes it

16. Be able to define the various clouds (nebulae) within the interstellar medium: dark nebulae, emission nebulae, and reflection nebulae

17. Know what protostars are and be able to explain how they form and evolve into main-sequence stars

18. Understand what happens during a star’s main-sequence lifetime

19. Know how a star leaves the main sequence and becomes a red giant

20. Be able to explain how observations of star clusters reveal how red giants evolve

21. Understand the significance of helium burning in a star and know the ways in which it begins

22. Be able to explain how planetary nebulae are created

23. Understand how white dwarfs are formed

24. Be able to explain the sequence of thermonuclear reactions that occur in the core and shells of an aging high-mass star

25. Understand how a high-mass star dies

26. Be able to explain how and why astronomers attempt to detect neutrinos from supernovae

27. Know what supernova remnants are

28. Know what a neutron star is and be able to describe its properties

29. Understand why the discovery of pulsars led to acceptance of the existence of neutron stars

30. Understand why pulsars slow down over time

31. Know what a millisecond pulsar is

32. Be able to state Einstein’s special theory of relativity
33. Understand how motion affects measurements of time and distance
34. Be able to state Einstein’s general theory of relativity
35. Understand the effects of gravity on light
36. Know how black holes form

2 Mathematical concepts

The following sections review, with examples, some of the more mathematical statements of the concepts that we used to understand Stars.

2.1 The Doppler Equation

Just as the perceived wavelength (pitch) of sound waves varies depending on whether their source is approaching you or receding from you, the perceived wavelength (color) of light depends on the relative motion of the source with respect to the observer. The shift in wavelength is given by

\[
\frac{\lambda_s - \lambda_o}{\lambda_o} = \frac{v}{c}
\]

\(\lambda_o\) is the original wavelength at which the light was emitted, also known as the rest wavelength. \(\lambda_s\) is the measured wavelength, which will be shifted from the original wavelength due to the motion of the source. \(v\) is the velocity with which the source is moving with respect to the observer and \(c\) is the velocity of light. \(v\) can be either positive or negative. A positive value of \(v\) corresponds to a source which is moving away from the observer. Negative values of \(v\) imply approach. If the source is moving away (\(v\) is positive) then \(\lambda_s\) must be larger than \(\lambda_o\). The observed wavelengths are longer than the emitted wavelengths. Such a shift to longer wavelengths is called a redshift regardless of the original wavelength of the light. When we talk about the spectrum we will often use the terms red and blue to indicate directions toward longer and shorter wavelengths respectively. Conversely, if a source of light is approaching, \(v\) is negative and the shifted wavelength, \(\lambda_s\), must be smaller than the original wavelength, \(\lambda_o\). This shift towards smaller wavelengths seen in approaching objects is called a blueshift, a shift in the direction of shorter wavelengths.

Example:
Q: The most prominent emission line of hydrogen gas occurs at a wavelength of 656.3nm. Suppose you observe this line in a distant galaxy at a wavelength of 660.0nm. Is the galaxy approaching or receding and how fast is it moving?

A: The original (emitted) wavelength is 656.3nm. The shifted (observed) wavelength is 660.0nm. Using the Doppler Equation

\[
\frac{660.0nm - 656.3nm}{656.3nm} = \frac{v}{c}
\]

\[0.0056 = \frac{v}{c}\]

\[v = 0.0056 \times c = 0.0056 \times 300,000 \text{ km/s} = 1691 \text{ km/s}\]

Since \(v\) is positive the galaxy is moving away from the Earth and Sun.

Q: A spectral line in an eclipsing binary star spectrum shifts by ±0.03 nanometer during the orbit. The line has a wavelength of 300 nanometer. What is the orbital velocity?

A: Using the Doppler formula, \(v = c\Delta \lambda / \lambda = 3 \times 10^5 \text{ km/s} \times 0.03/300 = 30 \text{ km/s}\).

### 2.2 Relationship between observed flux, temperature and radius of a star

This is a combination of geometry, the Stefan-Boltzmann law, and the inverse-square law of light. If you know a star’s temperature, the absolute luminosity of a star is the product of the surface area of the star and the emission per unit area from the star’s photosphere:

\[L = (4\pi R^2) \times \sigma T^4.\]

The apparent brightness \(B\), or flux, of the star is the energy observed per unit area in a detector or telescope in a given time. This also geometry; the absolute luminosity of the star is diluted by the total area of the sphere whose radius is the distance, \(d\), to the star:

\[B = \frac{L}{4\pi d^2}.\]

Putting the two together, we have:

\[B = \left(\frac{R}{d}\right)^2 \sigma T^4\]

\(B\) has ‘flux’ units: energy per area per time. Note the trends:
1. If one keeps the temperature the same, one observes a star of larger radius, the brightness increases (and vice versa).

2. If one somehow could find a sequence of stars with equal temperature but increasing radius, one could keep the brightness the same by putting stars with larger radii at larger distance $d$. In other words, keeping $R/d$ constant leaves $B$ constant. [A curiosity: $R/d$ is the angular size of the star. So keeping $R/d$ constant is equivalent to keeping the angular size of the star constant. We generally can’t measure the angular size of stars.]

3. If one keeps the radius and temperature the same, the brightness decreases with the square of the distance (just the inverse square law).

Example:

Q: Suppose a Star A is four times more distant than Star B but twice as hot. Their radii are the same. How do the brightness of the two stars compare?

A: Begin by taking the ratio of the above equation.

$$\frac{B_A}{B_B} = \frac{(R/d_A)^2 \sigma T_A^4}{(R/d_B)^2 \sigma T_B^4} = \left( \frac{d_B}{d_A} \right)^2 \left( \frac{T_A}{T_B} \right)^4$$

The questions states that $d_B/d_A = 1/4$ and $T_A/T_B = 2$. Therefore, $B_A/B_B = 1$: the overall brightness of the two stars is the same!

2.3 Relationship between observed flux, temperature and radius of a star

In class, we used that facts,

1. Stellar luminosity is proportional to stellar mass to the 3.5 power:

$$L \propto M^{3.5}$$

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1This is not the way it really works, as you know.
2. Main sequence lifetime (call it $\tau$) is proportional to stellar mass:

$$\tau \propto M/L$$

to find that

$$\tau \propto M^{-2.5}$$

or in terms of ratios

$$\frac{\tau}{\tau_\odot} = \left(\frac{M}{M_\odot}\right)^{-2.5}.$$ 

The main-sequence lifetime of the Sun, $\tau_\odot$, is roughly $1 \times 10^{10}$ years or 10 billion years.

**Example:**

Q: What is the lifetime of a $0.5M_\odot$ star?

A: $\tau = 1 \times 10^{10}(M/M_\odot)^{-2.5} = 1 \times 10^{10}(M/M_\odot)^{-2.5} = 1 \times 10^{10}(0.5)^{-2.5} = 5.7 \times 10^{10}$

years. We will see that this is longer than the age of the Universe.

### 2.4 Nuclear mass defect

The strong nuclear force overcomes the electrostatic repulsion of the positively charged protons to fuse. Recall that temperature is proportional to the average kinetic energy of gas particles. The higher the temperature, the better than chance that a particle can overcome the electrostatic repulsion. The easiest fusion reaction involves two protons (hydrogen nuclei) and becomes significant at temperatures of around 10 million K.

The **p-p chain** of reactions ultimately forms $^4$He from $^4$H. The atomic weights do not exactly match because the more exact weight of the proton is 1.0078 (in atomic mass units) and four of them add to 4.0312, while the weight of $^4$He is 4.0026, leaving a **mass defect** of 0.0286. This mass is converted to an amount of energy given by Einstein’s equation for the equivalence of mass and energy,

$$E = mc^2$$

where $c$ is the speed of light. Because the atomic mass unit is $1.66 \times 10^{-24}$ gm, the energy released by the conversion of four $^1$H nuclei to one $^4$He nucleus is

$$E = 0.0286 \times 1.66 \times 10^{-24} \times 9 \times 10^{20} = 4.3 \times 10^{-5}\text{erg}.$$ 

The percentage of mass converted to energy is $0.0286/4 \times 100 = 0.7\%$.

The largest mass defect is for iron; one must add energy to make nuclei more massive than iron. Therefore, one can continue “burn” nuclei by fusion until one reaches iron.

**Example:**

Q: Suppose we fuse a gram of hydrogen to helium, how much energy do we get?

A: *Since we know that 0.7% of the mass is converted to energy, we can use the mass-energy formula directly:*

\[
E = 0.007 \times 1 \text{ g} \times 3 \times 10^{10} \text{ cm/s} = 6.3 \times 10^{18} \text{ erg.}
\]

### 2.5 Angular momentum and rotation

All stars rotate, at least a bit. The star must conserve its angular momentum if it expands or contracts. The angular momentum is the sum of \( \mathbf{L} = mvr \) for each particle of mass \( m \) and speed \( v \) in the star. The distance \( r \) is the distance from the rotation axis.

Let’s apply the conservation of angular momentum to a simple case of a cloud of gas of radius 0.1 pc which collapses to form a star of one solar mass \((1 \text{ M}_{\odot})\). Let’s assume that the initial equatorial speed of the cloud is \( 1 \text{ km s}^{-1} \). We assume no mass loss so that the initial mass and final mass is the same. [NB: we discussed observational evidence for mass loss in class so this is large simplification.] Then:

\[
m_i v_i r_i = m_f v_f r_f
\]

where the subscripts \( i \) and \( f \) mean initial and final, respectively. Since \( m_i = m_f \), we can solve this for \( v_f \), say:

\[
v_f = v_i \left( \frac{r_i}{r_f} \right)
\]

\[
= 1 \text{ km s}^{-1} \left( \frac{3 \times 10^{17} \text{ cm}}{7 \times 10^{10} \text{ cm}} \right)
\]

\[
= 5 \times 10^5 \text{ km s}^{-1}.
\]

Note that this speed is larger than the speed of light (!!) which implies that the collapsing star must shed angular momentum during its formation.

### 2.6 Parallax

Measuring the distances to the stars. As the Earth orbits the Sun (moving 2 A.U. from one side of its orbit to the other) nearby stars appear to shift their position with respect to distant stars. The small angle through which the stars appear to move is called the parallax angle

\[
\text{distance} = \frac{1}{\text{parallax angle}} \quad d = \frac{1}{p}
\]

7
The distance \( (d) \) is measured in units of parsecs (pc) and the parallax angle \( (p) \) is measured in arc seconds. A star with a parallax angle of 1 arc second lies at a distance of 1 parsec = 3.26 light years. (You can check your understanding by working out the conversion between light years and parsecs from scratch.)

Examples:

Q: How far away is a star which has a parallax of 0.2 arc seconds?
A:
\[
d = \frac{1}{p} \quad d = \frac{1}{0.2} \quad d = 5 \text{ parsecs}
\]

Q: What is the parallax angle of a star at a distance of 10 parsecs?
A:
\[
d = \frac{1}{p} \quad 10 \text{ pc} = \frac{1}{p} \quad p = 0.1 \text{ arcseconds}
\]

2.7 Magnitudes

A measure of stellar brightness. In the Greek system the brightest stars were of the first magnitude. Stars that were not bright enough to be first magnitude but close were second magnitude stars. Stars fainter still were third magnitude and so on down to the faintest stars visible to the naked eye which were sixth magnitude stars. Modern day astronomers came up with the following rules which describe this system:

1. Fainter stars have numerically large magnitudes. The magnitudes of bright stars are small numbers or possibly even negative numbers. For example, an 11th magnitude star is fainter than a 9th magnitude star, and a \(-4\)th magnitude star is brighter than a 0th magnitude star.

2. Each magnitude corresponds to a difference in brightness of a factor of 2.512. More importantly, stars that differ in magnitude by 5 differ in brightness by a factor of 100 \((=2.512^5)\).

Modern astronomers also came up with a mathematical expression for these rules. Given two stars, label them ‘1’ and ‘2’, the difference in magnitude is:
\[
m_1 - m_2 = -2.5 \log_{10} \left( \frac{b_1}{b_2} \right)
\]
where \( b_1 \) and \( b_2 \) are the brightnesses of the two stars. By solving this equation for \( b_1/b_2 \), we see that this is exactly what we want:\(^2\)

\[
b_1/b_2 = 10^{-(m_1-m_2)/2.5} = 10^{0.4(m_2-m_1)} = 2.512^{(m_2-m_1)}.
\]

In words, for each magnitude that Star 2 is larger than Star 1, Star 1 is a factor of 2.512 brighter than Star 2.

**Examples:**

Q: Star A is 100 times brighter than Star B. Star B has a magnitude of 9. What is the magnitude of Star A?

A: *Since the difference in brightness is a factor of 100, the difference in magnitude must be 5. Since Star A is brighter than Star B its magnitude must be 5 numbers smaller that B’s magnitude. \( 9 - 5 = 4 \). Star A has a magnitude of 4.*

Q: What if Star A was 10,000 times brighter than Star B?

A: 10,000 = 100 \times 100. *Remember each factor of 100 corresponds to a difference in magnitude of 5. The total magnitude difference here then is \( 5 + 5 = 10 \), one 5 for each factor of 100. Star B has a magnitude of 9, so Star A has a magnitude of \( 9 - 10 = -1 \).*

Q: Star A is magnitude 2 and Star B has magnitude 7. What is the relative brightness between the two stars?

A: *Since each magnitude of difference between the two stars implies an factor of 2.512 in brightness, 5 magnitudes is \( 2.512^5 = 100 \). Since Star B has the large magnitude, it is 100 times fainter (less brightness) than Star A.*

### 2.8 Apparent vs. Absolute Magnitude

The discussion above points out that stars of equal luminosity can appear quite different if they are at unequal distances. We can either describe the brightness of a star in terms of how bright it appears to be, a quantity which depends on its distance, or we can choose a magnitude which tells us intrinsically how bright the star is. This other magnitude obviously cannot depend on the distance to the star and is called the star’s *absolute* magnitude. Astronomers have arbitrarily defined absolute magnitude so that

\(^2\)Remember, the logarithm is the answer to the following question: What is the number \( y \) so that \( x = 10^y \) ? We say: \( y = \log_{10} x \).
the absolute magnitude of a star is the magnitude it would have if it were placed at a
distance of 10 parsecs from the Earth. In contrast, the apparent magnitude of a star
is the just the magnitude a star appears to have as we see it in the sky. The apparent
magnitude of a star depends on its distance from us.

Examples:

Q: A star has an absolute magnitude of 4 and lies 1 parsec from the Earth. Suppose
that star is moved to a distance of 10 parsecs from the Sun. What is its absolute
magnitude?

A: *The absolute magnitude is still 4. Absolute magnitude does not depend on distance.*
*It measures the luminosity of the star.*

Q: A star has a parallax of 0.1 arc seconds and an apparent magnitude of 5. What is
its absolute magnitude?

A: *If the star has a parallax of 0.1 arc second it lies 10 parsecs from the Sun and
is already at the proper distance for measuring absolute magnitude. Its absolute
magnitude is therefore 5 as well.*

Q: A star has an apparent magnitude of 11 and lies 100 parsecs from the Sun. What
is its absolute magnitude?

A: *To determine the absolute magnitude we must move the star from 100 parsecs to
10 parsecs, ten times closer. The inverse square law says that the star will then
appear 100 times brighter (note that when you move objects closer they must appear
brighter by the factor you calculate using the inverse square law). If the star appears
100 times brighter its magnitude must decrease by 5. Thus when the star is at a
distance of 10 parsecs it will appear to have a magnitude of 11 – 5 = 6. It has an
absolute magnitude of 6.*

2.9 Distance modulus

We can consider the number of light rays from a source (like a star) to be fixed and
expanding away from the source like *surface* of an expanding balloon. The intensity
of the light is the number of beams per area. Since the surface area of a sphere is

\[ Area = 4\pi R^2 \]

where \( R \) is the radius of the sphere

\[ b = \frac{L}{4\pi R^2}. \]
Now take two identical stars but put them at different distances; the ratio of brightnesses will now be:

\[ \frac{b_1}{b_2} = \frac{R_2^2}{R_1^2}. \]

Using this in our mathematical definition for magnitudes, we can find an expression for the magnitude difference between the two identical stars at different distances:

\[ m_1 - m_2 = -2.5 \log_{10} \left( \frac{b_1}{b_2} \right) \]

which is what we had before, but now substituting relation above:

\[ = -2.5 \log_{10} \left( \frac{R_2^2}{R_1^2} \right) \]

and simplifying

\[ = 5 \log_{10} \left( \frac{R_1}{R_2} \right). \]

Finally, if we put Star 2 at \( R = 10 \text{pc} \), then \( M = m_2 \) and

\[ m_1 - M = 5 \log_{10} \left( \frac{R_1}{R_2} \right) - 5. \]

In words, the difference between the apparent magnitude and absolute magnitude of a star is proportional the logarithm of its distance. This difference in magnitude is called the “distance modulus”.

**Examples:**

Q: A star of known absolute magnitude has a distance modulus of 5. How far away is it?

A: Using the above relation gives \( 5 = 5 \log_{10} R - 5 \) or \( \log_{10} R = 2 \). Therefore, the star is at a distance of 100 pc.

Q: A star is 10 kiloparsec away. What is its distance modulus?

A: 10 kiloparsec is 10000 parsec (remember that kilo=1000). \( \log_{10}(10000) = 4 \), so the distance modulus is 15.

Q: Distance modulus is useful for estimating the magnitude of a star at a known distance. E.g. suppose that a star of known spectral type is found in a cluster with distance modulus of 12. The star has an absolute magnitude of \( M = 4 \). What do we expect its apparent magnitude to be?

A: 16 (direct use of distance modulus definition).
2.10 Escape velocity

**Escape velocity** is the speed required to leave a gravitational body so that it has zero speed at large distance. We can compute this velocity using energy conservation. Energy conservation says that the total energy of a system must remain the same. The two kinds of energy here are gravitational potential energy (the energy of *position*) and kinetic energy (the energy of *motion*). Kinetic energy is given by

\[ KE = \frac{1}{2}mv^2 \]

and gravitational potential energy is given by

\[ PE = -\frac{GMm}{r} \]

where \( m \) is the mass of projectile (space ship, whatever) and \( M \) is the mass of celestial body (Earth, Sun, black hole). The distance \( r \) is the distance from the center of the celestial body.

If the projectile is to have zero speed at a large distance, the total energy must be zero: \( E_{\text{far}} = KE + PE = 0 \). But by energy conservation \( E_{\text{surface}} = E_{\text{far}} = 0 \) so we have:

\[
\frac{1}{2}mv^2 - \frac{GMm}{r} = 0
\]

which we can solve for \( v \):

\[ v_{\text{escape}} = \sqrt{\frac{2GM}{r}}. \]

**Examples:**

Q: What is the escape velocity from the Earth?

A: *Just put in the appropriate values:*

\[
v_{\text{escape}} = \sqrt{\frac{2(6.67 \times 10^{-8})(6 \times 10^{27})}{(6.4 \times 10^8)}}
\]

\[
= 11 \text{ km s}^{-1}
\]

*I’m assuming cgs units in the first line and converting to km s\(^{-1}\) at the end.*

Q: What is the escape velocity from a 1 M\(_\odot\) black hole?
A: The speed of light, by definition! However, setting $v_{\text{escape}} = c$, we can use the formula for escape velocity to find the radius of this “surface”.

$$R_{\text{Sch}} = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-8})(2 \times 10^{33})}{(3.0 \times 10^{10})^2} = 3 \text{ km}$$

This expression for the size of the black hole is called the Schwarzschild radius which is why I called it $R_{\text{Sch}}$.

For additional review questions see the chapter ends in your textbook!