Name: SOLUTIONS

As we discussed in class, most angles encountered in astronomy are quite small so
degrees are often divided into 60 minutes and, if necessary, minutes in 60 seconds.
Therefore to convert an angle measured in degrees to minutes, multiply by 60. To
convert minutes to seconds, multiply by 60. To use trigonometric formulae, angles
might have to be written in terms of radians. Recall that $2\pi$ radians = 360 degrees.
Therefore, to convert degrees to radians, multiply by $2\pi/360$.

1. The average angular diameter of the Moon is 0.52 degrees. What is the angular
diameter of the moon in minutes?

   The goal here is to change units from degrees to minutes.

   $$0.52 \text{ degrees} \left(\frac{60 \text{ minutes}}{1 \text{ degree}}\right) = 31.2 \text{ minutes}$$

2. The mean angular diameter of the Sun is 32 minutes. What is the angular
diameter of the Sun in degrees?

   $$32 \text{ minutes} \left(\frac{1 \text{ degree}}{60 \text{ minutes}}\right) = 0.53 \text{ degrees}$$

   $$0.53 \text{ degrees} \left(\frac{2\pi}{360 \text{ degrees}}\right) = 0.0093 \text{ radians}$$

   Note that the angular diameter of the Sun is nearly the same as the angular
diameter of the Moon. This similarity explains why sometimes an eclipse of
the Sun by the Moon is total and sometimes is annular. See Chap. 3 for
more details.

3. Early astronomers measured the Sun’s physical diameter to be roughly 109 Earth
diameters (1 Earth diameter is 12,750 km). Calculate the average distance to the
Sun using trigonometry. (Hint: because the angular size is small, you can make
the approximation that $\sin \alpha = \alpha$ but don’t forget to express $\alpha$ in radians!).

   The small angle formula is: $D = d\alpha$ or $d = D\alpha$. We simply use the solution
from the previous problem for $\alpha$ with $D = 109 \times 12,750 \text{ km}$ to solve this one
as follows:

   $$d = 109 \times \frac{1.275 \times 10^4 \text{ km}}{0.0093 \text{ radians}} = 1.5 \times 10^8 \text{ km}$$
Suppose an asteroid’s closest approach to the Sun is 2 AU, and its greatest distance from the Sun is 4 AU. Sketch its orbit. How large is its semi-major axis, \( a \)? What is its period, \( P \)? What is the eccentricity of its orbit, \( e \)?

The first goal is to use the available information to solve the system for the semimajor axis. You can do this using the available information and the geometry of the ellipse: in particular, you know that the twice the distance between the Sun apocenter is the same as twice the distance between the other focus of the ellipse and pericenter. These conditions allow you to solve for the semimajor axis and then use Kepler’s third law to get the period.

\[
L/2 = 2ae + d_{peri} = d_{apo}
\]

\[2a = d_{peri} + d_{apo} \Rightarrow a = 3 \text{ AU}\]

In doing this, you have also solved for the distance between the center and the focus. In the third part, you can use this information to get the eccentricity \( e \) or the semi-major axis \( b \).

In class, I gave you that fact that the distance between the center and a focus: \( ae \). You can use this to get the eccentricity right away:

\[2ae = d_{apo} - d_{peri} \Rightarrow ae = 1 \text{ AU} \Rightarrow e = 1/3\]

Suppose you forgot this. You can still solve the problem. Call the distance between the center and a focus: \( q \). You can solve for \( q = ae \) as in the few lines above. Then by Pythagorean theorem:

\[b = \sqrt{h^2 + q^2} = \sqrt{(L/2 - q)^2 - q^2} = \sqrt{a^2 - q^2}\]

Solving this, gives \( q = ae \). (This is exactly how I solved this problem to get \( ae \) as the distance between the center and focus in the first place!)
Note that both of the following problems use Newton’s generalization of
Kepler’s Third Law (see Box 4-4 in your book):

\[ P^2 = \left( \frac{4\pi^2}{G(m_1 + m_2)} \right) a^3 \]

where \( P \) is in seconds, \( m \) is in kilograms and \( a \) is in meters if \( G = 6.67 \times 10^{-11} \)
N m\(^2\)/kg\(^2\).

5. Suppose the Sun were nine times as massive it now is, and the Earth’s orbit where
unchanged? Would the year be longer or shorter? By how many times? Explain
your answers.

Because the Earth’s orbit is unchanged, \( a \) is unchanged. Similarly the Earth
mass is so small compared to the Sun’s mass we can replace \( m_1 + m_2 \) by
\( M_{Sun} \). We can then use the technique of dividing Kepler’s third law for the
new situation by the same law for the original situation:

\[ \frac{P_{new}^2}{P_{orig}^2} = \left( \frac{\frac{4\pi^2}{G(9M_{Sun})}}{\frac{4\pi^2}{GM_{Sun}}} \right) a^3 \]

Cancelling similar terms in the numerator and denominator simplifies this
expression enormously:

\[ \frac{P_{new}^2}{P_{orig}^2} = \frac{1}{9} \]

or

\[ \frac{P_{new}}{P_{orig}} = \frac{1}{3} \]

This makes intuitive sense. If the Sun’s mass is increased, the attraction is
larger. Therefore the velocity of the Earth around the new more massive Sun
must increase to fall towards the Sun at rate which balances the attraction.
Therefore, the period must decrease.
A television satellite is in circular orbit about the Earth, with a period of 24 hours (as viewed from a fixed point in space). What is the distance from the Earth’s surface for such a satellite (express your answer in Earth radii)? If the satellite appears stationary to an earth-bound observer, what is the orientation of the satellite’s orbit relative to the Earth?

You probably know that the Space Shuttle orbits the Earth in roughly 1 hour. You could work this out directly from Newton’s generalization of Kepler’s Third Law:

\[ P^2 = \left( \frac{4\pi^2}{GM_{\text{Earth}}} \right) a^3 \]

having used the fact that a man-made satellite is much less massive than the Earth. Since the period \( P \) will increase as the distance from the Earth (semimajor axis) increases, we can find the distance \( a \) such that \( P \) is 24 hours by simply solving this equation for \( a \) after converting 24 hours to seconds:

\[ P^2 = \left( 24 \text{ hours} \times \frac{3600 \text{ sec}}{\text{hr}} \right)^2 = \left[ \frac{4\pi^2}{6.67 \times 10^{-11} \times 5.974 \times 10^{24}} \right] a^3 \]

or

\[ a = 4.22 \times 10^7 \text{ m} = \frac{4.22 \times 10^7 \text{ m}}{1.276 \times 10^7 / 2 \text{ m}} = 6.6 \text{ Earth radii}. \]

This orbit is 24 hours no matter what the orientation of this orbit relative to earth. But, of course, it will only appear stationary to an earth observer if it orbits in the plane of the Earth’s equator.

Finally, note that the semi-major axis \( a \) is measured from the center of the Earth. So, the height of the satellite above the surface of the Earth is 6.6 – 1.0 = 5.6 Earth radii.