A850: Exercise #4  
Hamiltonian Dynamics

Due: 22 Mar 2011

1 Generating functions
Derive the canonical transformations for generating functions of Types 1, 3 and 4. [Hint: Recall that we did Type 2 in class.]

2 The Double Pendulum
The double pendulum consists of a mass $m_1$ suspended by a rod of length $l_1$, from which is suspended a second mass $m_2$ and rod of length $l_2$. The apparatus is confined to a motion in a vertical plane (see Fig. 1).

(a) Write down the Lagrangian for the system and obtain the equations of motion for $\theta_1$ and $\theta_2$.
(b) Write down the two-dimensional oscillation equations (assuming of course that $|\theta_1|, |\theta_2| \ll 1$).
(c) Assume that $l \equiv l_1 = l_2$ and find the eigenfrequencies and eigenvectors. Describe the motion of each degree of freedom.

3 Liouville equation and the pendulum
Consider the pendulum of length $l$ and mass $m$ in a gravitational field $g$. The kinetic energy and potential energy are

$$T = \frac{1}{2} ml^2 \dot{\theta}^2$$

and

$$V = mgl (1 - \cos \theta).$$

(a) Derive the Hamiltonian and write down Hamilton’s equations.
(b) Transform these equations to a non-dimensional set of variables \((q, p) \rightarrow (z_1, z_2)\) such that \(\tau = \omega t\) where \(\omega^2 = g/l\) and the dimensionless energy is:

\[
\varepsilon \equiv \frac{E}{mgl} = \frac{1}{2} z_2^2 + (1 - \cos z_1).
\]

Show that \(\tau_0 = \frac{2\pi}{\omega}\) is the period of small oscillations for this system.

(c) Sketch the phase portrait in these units, paying special attention to the three domains: 1) \(\varepsilon < 2\), the oscillation or libration regime; 2) \(\varepsilon > 2\), the rotation regime; and 3) \(\varepsilon = 2\), the critical infinite period, the separatrix.

(d) Consider a disk-like domain of initial configurations \(U_s\) enclosed within the circle centered at \((q, p) = (z_1, z_2) = (0, 1)\) with radius of 1/2 in these units. See Fig. 2 for a plot of these initial conditions. Compute the evolution of the disk from \(\tau = 0\) to each of \(\tau = 0.25\tau_0, 0.5\tau_0, \text{ and } \tau_0\).

Remarks:

- Argue that it is sufficient to compute the trajectories on the circle at edge of the disk. In other words, a point on the inside of the disk can not pass through a boundary point.
You may solve this any way you like. E.g. there is an analytic solution in terms of elliptic integrals. But I recommend an numerical solution, using your RK4 routine to solve the equations of motion for say 32 points equally spaced around the circle at the perimeter of $U_s$ at $\tau = 0$.

For insight consider this problem in the small angle limit. In this limit, the solution is analytic. If you have any worries about your solution, make sure it agrees with the small angle limit.

(e) Repeat the experiment but now with the circle centered at $(q, p) = (0, 1.5)$. Note: some points will be on or near the separatrix. Compute the evolution of the disk for $\tau = 0, 0.2\tau_0, 0.4\tau_0$ and $0.75\tau_0$. [NB: the “evolution” at $\tau = 0$ is the initial condition, of course.]

(f) Repeat the experiment but now with the circle centered at $(q, p) = (0, 2)$. Compute the evolution at times $\tau = 0, 0.1\tau_0, 0.25\tau_0$ and $0.5\tau_0$.

(g) For each of the three experiment with different disk centers, make a plot the disk at each of these times in the $(q, p)$ plane (e.g. a phase portrait). Interpret the results. Is Liouville’s theorem obeyed?