Stellar Modeling

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1 The Equations of Stellar Structure

We consider the modeling of stars in hydrostatic and thermal equilibrium (i.e., time-dependent processes are ignored for now). From the modeling, we hope to know how such global properties as the luminosity and radius of a star depend on its mass and initial chemical composition. To do so, we need to solve a set of four differential equations as we discussed in Chapter 1, which are equivalent to a fourth-order differential equation:

The mass conservation

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}, \tag{1}$$

hydrostatic equation

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}, \tag{2}$$

energy equation

$$\frac{dL_r}{dM_r} = \epsilon, \tag{3}$$

and the heat transfer method (radiative, conduction, and/or convection), or the form of the model

$$\nabla \equiv \frac{d\ln T}{d\ln P}. \tag{4}$$

To implement the specific heat transfer, compute

$$\nabla_{rad} = \frac{3}{16\pi ac}\frac{P\kappa}{T^4}\frac{L_r}{GM_r}, \tag{5}$$

which assumes that the transfer is all due to the radiation (the conduction is neglected here). Then we may set

$$\nabla = \nabla_{rad} \quad \text{if} \quad \nabla_{rad} \leq \nabla_{ad} \tag{6}$$

for pure diffusive radiative transfer or conduction, or

$$\nabla = \nabla_{ad} \quad \text{if} \quad \nabla_{rad} > \nabla_{ad} \tag{7}$$

when adiabatic convection is present locally — as in a mixing length theory.

Four boundary conditions are required to close the system. For simplicity, we choose "zero" conditions which are $r = L_r = 0$ at the center ($M_r = 0$), and $\rho = T = 0$ at the surface ($M_r = M$). Here M was specified beforehand.

Of course, we assume that we already know the microscopic constituent physics, as we have discussed; i.e., the quantities P, E, κ, and ϵ as functions of ρ, T, and X, where X is shorthand for composition. These quantities should be available on demand either in analytic and numerical forms.

It should be noted that the availability of the quantities does not guarantee that a solution to the equations always exists or unique!

Before going into how the above stellar structure equations are solved in practice, we first consider a simplified case, which is both useful practically and illumining.

2 Polytropic Equations of State and Polytropes

We define a polytropic stellar model (or polytrope) to be one in which the pressure is given by

$$P(r) = K\rho^{1+1/n}(r) \tag{8}$$

where both the *polytropic index* n and K are constant. This model allows us to avoid dealing with both the heat transfer and thermal balance.

We have encountered such power law before; e.g., the EoS for a zero temperature, completely degenerate electron gas (e.g., $P_e = 1.0 \times 10^{13}(\frac{\rho}{\mu_e})^{5/3}$ dyn cm^{-2} if non-relativistic). The polytrope is also a good approximation for certain types of adiabatic convection zones. For a region with efficient convection, i.e., $\nabla = \nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{ad} = 1 - 1/\Gamma_2$. If Γ_2 is assumed constant, then

$$P(r) \propto T^{\Gamma_2/(\Gamma_2-1)}(r). \tag{9}$$

If in addition, the gas is ideal, then $P(r) \propto \rho^{\Gamma_2}(r)$.

For a polytrope, we can derive from the hydrostatic and mass conservation equations (in Euclidean coordinates) the following

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -\frac{G}{r^2}\frac{dM_r}{dr} = -4\pi G\rho \tag{10}$$

Now perform the transformations to make the the equation dimensionless: $\rho(r) = \rho_c\theta^n(r)$ and $r = r_n\xi$, we then have the *Lane-Emden equation*:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_n}{d\xi}\right) = -\theta_n^n \tag{11}$$

where P_c is defined (from the EoS) as $P_c = K\rho_c^{1+1/n}$ and $r_n^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2}$. The solutions are called "Lane-Emden solutions" and denoted by $\theta_n(\xi)$.

Note that if the EoS for the model material is an ideal gas with constant μ, θ_n measures temperature $T(t) = T_c\theta_n(r)$, where $T_c = K\rho_c^{1/n}(N_A k/\mu)^{-1}$.

Since the *Lane-Emden equation* is a second-order differential equation, we need two real boundary conditions: First for ρ_c to really be the central density, we require that $\theta_n(\xi = 0) = 1$; Second, the spherical symmetry at the center (dP/dr vanishing at $r = 0$) requires that $\theta'_n(\xi = 0) = 0$, with the resulting regular solutions called "E-solutions" (the abandoned divergent solutions at center may be used in part of a star, however).

The surface of a model star is where the first zero of θ_n occurs, $\theta_n(\xi_1) = 0$, where ξ_1 is the location of the surface. This interpretation of the solution is not a boundary condition.

Analytical E-solutions for θ_n are obtainable for $n = 0, 1,$ and 5, as given in the textbook. For example, for $n = 0$, $\rho(r) = \rho_c$. This constant-density sphere has the solution (when the boundary conditions at $\xi = 0$ are used):

$$\theta_0(\xi) = 1 - \frac{\xi^2}{6}. \tag{12}$$

Clearly, $\xi_1 = 6^{1/2}$.

Numerical methods must be used for general n. So given n and K, we can in principle find the dependence of P and ρ on ξ. However, to get the absolute physical numbers, we need $R = r_n\xi_1$, which depends on ρ_c, as shown above. These two parameters are linked by the stellar mass, which we wish to specify via

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho(r) dr \\ &= 4\pi r_n^3 \rho_c \int_0^{\xi_1} \xi^2 \theta_n^n d\xi \\ &= 4\pi r_n^3 \rho_c (-\xi^2 \theta'_n)_{\xi_1} \end{aligned} \tag{13}$$

For a given n, $(-\xi^2\theta'_n)_{\xi_1}$ is known, the above equation then gives ρ_c, and hence P_c, in terms of M.

A useful quantity that depends only on n is the ratio

$$\frac{\rho_c}{<\rho>} = \frac{1}{3}\left(\frac{\xi}{-\theta'_n}\right)_{\xi_1}, \tag{14}$$

where $<\rho>$ is the volume-averaged mean density of a star. For the pressure of a completely degenerate, but non-relativistic electron gas ($\propto \rho^{5/3}$), $n = 1.5$. For the completely degenerate and fully relativistic case, $n = 3$. For an ideal gas convection zone ($\propto \rho^{5/3}$), $n = 1.5$. Unfortunately, neither of these values have analytic E-functions.

3 The Eddington Standard Model

This gives a simple example of the use of polytropes in making a stellar pseudo-model, which approximately incorporates the energy and radiative transfer equations.

Recall that in case of no convection the radiative transfer equation can be expressed as

$$\nabla = \frac{3}{16\pi ac}\frac{P\kappa}{T^4}\frac{L_r}{GM_r}, \tag{15}$$

where

$$\nabla \equiv \frac{d\ln T}{d\ln P} = \frac{1}{4}\frac{P}{P_{rad}}\frac{dP_{rad}}{dP}. \tag{16}$$

Here we have introduced the radiative pressure $P_{rad} = aT^4/3$ so that T can be replaced to get

$$\frac{dP_{rad}}{dP} = \frac{\kappa L_r}{4\pi cGM_r}. \tag{17}$$

We define

$$< \varepsilon(r) > \equiv \frac{L_r}{M_r} = \frac{\int_0^r \varepsilon dM_r}{\int_0^r dM_r} \tag{18}$$

where the energy equation $dL_r/dM_r = \varepsilon$ is used. We further define

$$< \eta(r) > \equiv \frac{< \varepsilon(r) >}{< \varepsilon(R) >} = \frac{L_r/M_r}{L/M}. \tag{19}$$

The transfer equation then becomes

$$\frac{dP_{rad}}{dP} = \frac{L}{4\pi cGM}\kappa(r)\eta(r). \tag{20}$$

Assuming that the surface pressure is equal to zero, the integration of the above equation gives

$$P_{rad}(r) = \frac{L}{4\pi cGM} < \kappa(r)\eta(r) > P(r) \tag{21}$$

where the average expression is

$$< \kappa(r)\eta(r) > = \frac{1}{P(r)}\int_0^{P(r)} \kappa\eta dP. \tag{22}$$

If we can assume that $< \kappa(r)\eta(r) >$ varies weakly with position in a star, or close to a constant, as Eddington did, then the ratio of $1 - \beta \equiv P_{rad}/P$ is a constant and so is β. This constancy may be translated into a T vs. ρ relation as follows. If the pressure is contributed by ideal gas plus radiation only, then

$$P_{rad} = P - P_{gas} = (1/\beta - 1)P_{gas} = \frac{1-\beta}{\beta}\frac{N_A k}{\mu}\rho T = aT^4/3 \tag{23}$$

$$T(r) = \left(\frac{1-\beta}{\beta} \frac{3}{a} \frac{N_A k}{\mu} \right)^{1/3} \rho^{1/3}(r). \tag{24}$$

$$P = \frac{P_{gas}}{\beta} = \frac{N_A k}{\mu} \frac{\rho T}{\beta} = K \rho^{4/3}(r), \tag{25}$$

where

$$K = \left[\frac{1-\beta}{\beta^4} \frac{3}{a} \left(\frac{N_A k}{\mu} \right)^4 \right]^{1/3}. \tag{26}$$

So we have a polytrope with $n = 3$, which can be readily solved numerically.

4 Numerical calculation of the Lane-Emden equation

A convenient way to numerically solve such an equation is to cast the second-order problem in the form of two first-order equations by introducing the new variables $x = \xi$, $y = \theta_n$, and $z = (d\theta_n/d\xi) = (dy/dx)$:

$$\begin{aligned} y' &= \frac{dy}{dx} = z \\ z' &= \frac{dz}{dx} = -y^n - \frac{2}{x} z \end{aligned} \tag{27}$$

Here we use a simple "shooting method", whereby one "shoots" from a starting point and hopes that the shot will end up at the right place; e.g., using a "Runge-Kutta" integrator. The solution is "leap-frogged" from x to $x + h$, where h is called the "step size". Suppose we know the values of y and z at some point x_i and call these values y_i and z_i. If h is some carefully chosen, then we can use the above equations to find y_{i+1} and z_{i+1} at $x_{i+1} = x_i + h$.

Care needs to be taken at the origin, where z' is indeterminate because both x and z are equal to zero. The resolution to this problem is to expand $\theta_n(\xi)$ in the Lane-Emden equation in a series about the origin. Inserting $\theta_n(\xi) = a_0 + a_1\xi + a_2\xi^2 ...$ into the equation, compare the coefficients of individual ξ terms, and apply the boundary condition to establish the constants in the expansion, we get

$$\theta_n(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 ... \tag{28}$$

For $\xi \to 0$, find that $z' \to -1/3$, which may be used to start the integration.

This way, the calculation may march from the origin to the surface, when $y = \theta_n$ cross the zero.

5 Newton-Raphson and Henyey Methods

A more powerful technique to solve the stellar structure equations is the "integration" over the model, instead of shooting from one point to another.

Consider a second-order system as an example

$$\begin{aligned} \frac{dy}{dx} &= f(x, y, z) \\ \frac{dz}{dx} &= g(x, y, z) \end{aligned} \tag{29}$$

with boundary conditions on y and z specified at the endpoints of the interval $x_1 \leq x \leq x_N$ and may be generally expressed as

$$\begin{aligned} b_1(x_1, y_1, z_1) &= 0 \\ b_N(x_N, y_N, z_N) &= 0 \end{aligned} \tag{30}$$

where y_i and z_i are $y(x_i)$ and $z(x_i)$.

Assuming that f, g, b_1, and b_2 are well behaved, the differential equations can be cast in a "finite difference form" over a predetermined "mesh" in x; i.e., $x_1, x_2, ..., x_N$ at which y and z are to be evaluated. For simplicity, consider that the mesh interval is constant; i.e., $x_{i+1} - x_i = \Delta x$ for all i. The equations can hen be expressed as

$$\begin{aligned} \frac{y_{i+1} - y_i}{\Delta x} &= \frac{1}{2}(f_{i+1} + f_i) \\ \frac{z_{i+1} - z_i}{\Delta x} &= \frac{1}{2}(g_{i+1} + g_i) \end{aligned} \tag{31}$$

where f_i, for example, is shorthand for the function $f(x_i, y_i, z_i)$. The above expressions then represent $2N - 2$ equations, which together with the two boundary conditions can in principle be used to solve $2N$ variables y_i and z_i. Thus, this is *not* an initial value problem. However, the difficulty is that the variables are all mixed up among the equations, generally in a nonlinear fashion.

5.1 Newton-Raphson Method

One way to get out of this difficulty is to use the Newton-Raphson method: find the solution by linearizing the equations and the boundary conditions.

Suppose that we have a "guessed" solution (e.g., from the shooting method) that gives y_i and z_i for all i, which generally do not satisfy the equations. we may make corrections

$$\begin{aligned} y_i &\rightarrow y_i + \Delta y_i \\ z_i &\rightarrow z_i + \Delta z_i \end{aligned} \tag{32}$$

so that the new y_i and z_i might satisfy both the equations and the boundary conditions. We now *estimate* the values of Δy_i and Δz_i for all i by letting the new y_i and z_i satisfy the linearized equations and boundary conditions. For example, the first equation of Eq. 31 becomes

$$\begin{aligned} y_{i+1} + \Delta y_{i+1} - y_i - \Delta y_i = \\ \frac{\Delta x}{2}\left[f_{i+1} + \left(\frac{\partial f}{\partial y}\right)_{i+1} \Delta y_{i+1} + \left(\frac{\partial f}{\partial z}\right)_{i+1} \Delta z_{i+1}\right] \\ + \frac{\Delta x}{2}\left[f_i + \left(\frac{\partial f}{\partial y}\right)_i \Delta y_i + \left(\frac{\partial f}{\partial z}\right)_i \Delta z_i\right] \end{aligned} \tag{33}$$

Some manipulation leads to

$$\begin{aligned} y_{i+1} - y_i - \frac{\Delta x}{2}(f_{i+1} + f_i) = \\ \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_i + 1\right] \Delta y_i + \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_{i+1} - 1\right] \Delta y_{i+1} \\ + \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_i\right] \Delta z_i + \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_{i+1}\right] \Delta z_{i+1} \end{aligned} \tag{34}$$

Note that the left-hand side of these equations are zero when the difference equations are satisfied; that is when Δy_i and Δz_i go to zero. Similarly, the boundary conditions can be linearized into

$$b_{(1 \text{ or } N)} + \left(\frac{\partial b}{\partial y}\right)_{(1 \text{ or } N)} \Delta y_{(1 \text{ or } N)} + \left(\frac{\partial b}{\partial z}\right)_{(1 \text{ or } N)} \Delta z_{(1 \text{ or } N)} = 0. \tag{35}$$

We can arrange all these equations in a matrix form

$$\mathbf{M} \cdot \mathbf{U} = \mathbf{R}. \tag{36}$$

in which,

$$\mathbf{U} \equiv (\Delta y_1, \Delta z_1, \Delta y_2, \Delta z_2, \cdots, \Delta y_N, \Delta z_N)^{\mathbf{T}} \tag{37}$$

where the superscript "T" indicates transpose;

$$\mathbf{R} = (-b_1, Y_{3/2}, Z_{3/2}, \cdots, Y_{N-1/2}, Z_{N-1/2}, -b_N)^{\mathbf{T}} \tag{38}$$

where

$$\begin{aligned} Y_{i+1/2} &\equiv y_{i+1} - y_i - \frac{\Delta x}{2}(f_{i+1} + f_i) \\ Z_{i+1/2} &\equiv z_{i+1} - z_i - \frac{\Delta x}{2}(g_{i+1} + g_i); \end{aligned} \tag{39}$$

and finally

$$\left|\begin{array}{cccccccc}
\left(\frac{\partial b}{\partial y}\right)_1 & \left(\frac{\partial b}{\partial z}\right)_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
A_1+1 & B_1 & A_2-1 & B_2 & 0 & 0 & \cdots & \cdots \\
C_1 & D_1+1 & C_2 & D_2-1 & 0 & 0 & \cdots & \cdots \\
0 & 0 & A_2+1 & B_2 & A_3-1 & B_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \left(\frac{\partial b}{\partial y}\right)_N & \left(\frac{\partial b}{\partial z}\right)_N
\end{array}\right| \tag{40}$$

where

$$\begin{aligned}
A_i &\equiv \frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_i, \quad C_i \equiv \frac{\Delta x}{2}\left(\frac{\partial g}{\partial y}\right)_i \\
B_i &\equiv \frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_i, \quad D_i \equiv \frac{\Delta x}{2}\left(\frac{\partial g}{\partial z}\right)_i.
\end{aligned} \tag{41}$$

Once the solution set $\mathbf{U}$ is found, then new values of y_i and z_i are obtained by adding Δy_i and Δz_i to the corresponding old guesses. If all goes well, then the corrections decrease as the square of their absolute values. We iterate the above procedure until Δy_i and Δz_i become sufficiently small.

5.2 Eigenvalue Problems and the Henyey Method

However the above scheme needs to be slightly modified if the location of the boundary is not known before-hand. In the polytropic stellar structure we want to get, the radius ξ_1 needs to be found as part of the solution. To do so, we do a simple conversion,

$$x \to x = \xi/\lambda \tag{42}$$

where $\lambda = \xi_1$. After this conversion, x is within the closed interval $[0, 1]$ and can be divided into a grid, upon which the difference equations and boundary conditions can be applied. The converted f and g are generally also depend on λ; e.g., the Lane-Emden equation becomes

$$\begin{aligned}
y' &= \frac{dy}{dx} = \lambda z \\
z' &= \frac{dz}{dx} = -\lambda y^n - \frac{2}{x} z.
\end{aligned} \tag{43}$$

Now we have one more parameter λ, or an *eigenvalue*, to determine. This can be done since we have one more boundary condition, $y = 0$ at the new $x = 1$, in addition to the two at the center $(y = 1, z = 0)$.

To get the solution, we just need to add the correction $\lambda \to \lambda + \Delta\lambda$ in the linearization of f, g, and the three boundary conditions; e.g.,

$$f \to f + \left(\frac{\partial f}{\partial y}\right)_{(z_i,\lambda)} \Delta y_i + \left(\frac{\partial f}{\partial z}\right)_{(y_i,\lambda)} \Delta z_i + \left(\frac{\partial f}{\partial \lambda}\right)_{(y_i,z_i)} \Delta\lambda. \qquad (44)$$

Each difference equation thus contributes terms in $\Delta\lambda$ to the matrix algebra problem; i.e., there is now an extra row in the main matrix corresponding to the additional boundary condition and an extra column for the extra unknown, $\Delta\lambda$.

6 The Structure of the Envelope

To construct a more realistic stellar model, we need to deal with the surface layer more carefully. Starting from the surface inward, we may model the atmosphere as described in the heat transfer chapter, where we have assumed that convection plays no role in hear transport between the true and photosphere surface, consistent with our notion of a radiating, static, and visible surface (although this is not true for the sun). In such a radiative heat transfer region, $\nabla = \nabla_{rad}$ with

$$\nabla = \frac{d\ln T}{d\ln P} = \frac{3\kappa L}{16\pi acGM}\frac{P}{T^4} = \frac{1}{(K')^{n+1}(1+n_{eff})}P^{n+1}T^{-(n+s+4)}, \qquad (45)$$

in which

$$K' = \left(\frac{1}{1+n_{eff}}\frac{16\pi acGM}{3\kappa_g L}\right)^{1/(n+1)}, \qquad (46)$$

where $n_{eff} = (s+3)/(n+1)$ is the "effective polytropic index" and the opacity has been written in the interpolation form for ideal gas, $\kappa = \kappa_g P^n T^{-n-s}$. At the photosphere, we have

$$\nabla_p = \frac{1}{(K')^{n+1}(1+n_{eff})}P_p^{n+1}T_p^{-(n+s+4)}. \qquad (47)$$

It is easy to show $\nabla_p = 1/8$, using $P_p = 2g_s/3\kappa_p$, $g_s = GM/R^2$, and $L = 4\pi R^2\sigma T_{eff}^4$ at the photosphere, as well as $\sigma = ac/4$.

Next we consider the stellar "envelope", which consists of the portion of a star that starts at the photosphere, and continues inward, but contains negligible mass, has no thermonuclear or gravitational energy sources, and is in hydrostatic equilibrium. We want to see how deep the radiative envelope may be until the convection takes over for heat transfer. In the radiative heat transfer region, Eq. 45 can be re-written as

$$P^n dP = (1+n_{eff})(K')^{n+1}T^{n+s+3}dT. \qquad (48)$$

If $n+s+4$ is non-zero, this can be integrated to give

$$P^{n+1} - P_p^{n+1} = (K')^{n+1}(T^{n+s+4} - T_p^{n+s+4}). \qquad (49)$$

Replacing P and P_p with ∇ and ∇_p from Eqs. 45 and 47, respectively, the above equation can be expressed as

$$\nabla = \frac{1}{1+n_{eff}} + \left(\frac{T_p}{T}\right)^{n+s+4}\left(\nabla_p - \frac{1}{1+n_{eff}}\right) \tag{50}$$

In case that $n+1$ and $n+s+4$ are both positive and as we go to deep depths in the envelope, then we have

$$P \rightarrow K' T^{1+n_{eff}} \tag{51}$$

and

$$\nabla \rightarrow \frac{1}{1+n_{eff}}. \tag{52}$$

Thus as far as the interior structure is concerned, we could just as well have used zero boundary conditions for the density and temperature, as was done above. Examples for this case include Kramer's opacity ($n = 1$ and $s = 3.5$) and electron scattering ($n = s = 0$). For ideal gas with constant composition, $P = K\rho^{1+1/n_{eff}}$, where $K = \left(\frac{N_A k}{\mu}\right)^{1+1/n}(K')^{-1/n}$. We can then have a polytropic-like solution in the envelope, though it does not need to be of the complete E-solution variety. With appropriate conditions of continuity, one can then connect the envelope solution to the interior one. Specifically, if Kramers' opacity holds in the envelope, then $n_{eff} = 3.25$ and $\nabla = 0.2353 < \nabla_{ad} = 1 - 1/\Gamma_2 = 0.4$, which implies no convection as we have assumed. The same is true for the electron scattering opacity.

6.1 Radiative envelope structure

Assuming that the envelope is radiative with a constant n_{eff}, as in Eq. 52, we may determine the temperature distribution as a function of the radius. Rewrite the equation of hydrostatic equilibrium in the form

$$\frac{dP}{dr} = \frac{P}{\nabla}\frac{1}{T}\frac{dT}{dr} = -\frac{GM}{r^2}\rho. \tag{53}$$

Using $P = \rho N_A kT/\mu$ to replace the pressure, we have

$$(1+n_{eff})\frac{dT}{dr} = -\frac{GM\mu}{N_A k r^2}. \tag{54}$$

An integration of this gives

$$\begin{aligned} T(r) =& \frac{1}{1+n_{eff}}\frac{GM\mu}{N_A k}\left(\frac{1}{r} - \frac{1}{R}\right) \\ =& \frac{2.3\times 10^7\ \mathrm{K}}{1+n_{eff}}\mu\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}\left(\frac{1}{x} - 1\right) \end{aligned} \tag{55}$$

where $x = r/R$. It is clear that for a reasonable value of n_{eff} (e.g., $= 3.25$ for Kramers' opacity) and μ (≈ 0.6 for a Pop I star), the last term of the r.h.s of the above equation has to be very small; i.e., the temperature drops to a photosphere temperature over a radius $\delta r \lesssim$ 1% of R. We can also get a density distribution from the above temperature distribution, using $P = K'T^{1+n_{eff}}$ for ideal gas. It can be shown that for a solar mass and luminosity, for example, traversing 15% of the total radius inward from the surface uses up only a little less than 1% of the mass, confirming our assumption that $M_r \approx M$ through the envelope.

6.2 Convective envelopes and stars

An important counterexample to the above case is where the envelope opacity is due to H^- ($n = 1/2$, and $s = -9$) and photosphere boundary conditions have a strong influence on the underlying layers. It is also true that in cool stars, where H^- opacity is important, the underlying layers are convective and the above analysis does not apply. We first check where the convection may take place in a such envelope.

For H^- opacity (hence $n_{eff} = -4$), Eq. 50 can be expressed as

$$\nabla(r) = -\frac{1}{3} + \frac{11}{24}\left(\frac{T}{T_{eff}}\right)^{-9/2} \tag{56}$$

where $\nabla_p = 1/8$ is inserted. Note that since temperature increases with depth, so does ∇. Eventually, when $\nabla > \nabla_{ad}$, the stellar material becomes convective. For simplicity, we assume that the convection is adiabatic. Thus at depths deeper than the critical depth, $\nabla = \nabla_{ad} = 0.4$ for ideal gas ($\Gamma_2 = 5/3$). The transition to convection occurs at $T_f = (8/5)^{2/9}T_{eff} = 1.11T_{eff}$. The corresponding pressure $P_f = 2^{2/3}P_p$ can be found from

$$\left(\frac{P}{P_p}\right)^{n+1} = 1 + \frac{1}{1+n_{eff}}\frac{1}{\nabla_p}\left[\left(\frac{T}{T_p}\right)^{n+s+4} - 1\right], \tag{57}$$

which can be easily obtained from Eq. 49.

For the assumed convection, the implying polytrope of index is 3/2 and

$$P = K'_{n=3/2}T^{5/2}. \tag{58}$$

For a completely convective star, $K'_{n=3/2}$ can be related to the the mass and radius of the star as defined by the E-solution polytrope

$$K'_{n=3/2} \propto \frac{1}{\mu^{5/2}M^{1/2}R^{3/2}} \tag{59}$$

Using $P_f = K'_{n=3/2}T_f^{5/2}$ and $L = 4\pi\sigma R^2 T_{eff}^4$ to replace R, we get

$$T_{eff} \propto \mu^{13/51}(M/M_{\odot})^{7/51}(L/L_{\odot})^{1/102} \tag{60}$$

Therefore, such a completely convective star has a nearly constant effective temperature, independent of the luminosity.

In reality, ionization processes and convection, albeit almost negligible, occur in the outer layers of nearly all stars and a complete and accurate integration including all effects is necessary in modeling real stars.

7 Review

Key concepts: Polytropes, Lane-Emden equation, the Eddington Standard Model, Newton-Raphson and Henyey Methods

What are the basic equations and the boundary conditions that are needed to construct a normal stellar interior model? What microscopic physics should be implemented in such a modeling?

Why is the polytropic stellar model only a second-order differential equation?

Please give two examples of the situations in which a polytropic equation of state may be used?

What is the key assumption made in the Eddington Standard Model? Why may this assumption be reasonable?

What is the basic approach of the Newton-Raphson or Henyey Method in solving the stellar equations? How is it different from a simply "shooting method"?