Orbits in Axisymmetric Potentials I

Axisymmetric potentials (oblate or prolate) are far more realistic examples to consider in astronomy. Elliptical galaxies might well be spheroidal (but could also be ellipsoidal), while disk galaxies almost certainly are axisymmetric (though highly flattened).

For axisymmetric systems the coordinate system of choice are cylindrical coordinates \((R, \phi, z)\), and \(\Phi = \Phi(R, z)\).

Solving Newton’s equations of motion in cylindrical coordinates yields:

\[
\begin{align*}
\ddot{R} - R\dot{\phi}^2 &= -\frac{\partial \Phi}{\partial R} \\
\frac{d}{dt} \left( R^2 \dot{\phi} \right) &= 0 \\
\ddot{z} &= -\frac{\partial \Phi}{\partial z}
\end{align*}
\]

The second of these expresses conservation of the component of angular momentum about the \(z\)-axis; \(L_z = R^2 \dot{\phi}\), while the other two equations describe the coupled oscillations in the \(R\) and \(z\)-directions.

**NOTE:** For stars confined to the equatorial plane \(z = 0\), the equations of motion are identical to that of motion in a spherical density distribution (not surprising, since in this case the motion is once again central). Therefore, orbits confined to the equatorial plane are rosette orbits.
Orbits in Axisymmetric Potentials II

As for the spherical case, we can reduce the equations of motion to

\[
\begin{align*}
\ddot{R} &= -\frac{\partial \Phi_{\text{eff}}}{\partial R} \\
\ddot{z} &= -\frac{\partial \Phi_{\text{eff}}}{\partial z}
\end{align*}
\]

with \( \Phi_{\text{eff}}(R, z) = \Phi(R, z) + \frac{L_z^2}{2R^2} \) the effective potential. The \( \frac{L_z^2}{R^2} \)-term serves as a centrifugal barrier, only allowing orbits with \( L_z = 0 \) near the symmetry-axis.

This allows us to reduce the 3D motion to 2D motion in a Meridional Plane \((R, z)\), which rotates non-uniformly around the symmetry axis according to \( \dot{\phi} = L_z/R^2 \).

In addition to simplifying the problem, it also allows the use of surfaces-of-section to investigate the orbital properties.

For the energy we can write

\[
E = \frac{1}{2} \left[ \dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2 \right] + \Phi = \frac{1}{2} \left( \dot{R}^2 + \dot{z}^2 \right) + \Phi_{\text{eff}}
\]

so that the orbit is restricted to the area in the meridional plane satisfying \( E \geq \Phi_{\text{eff}} \). The curve bounding this area is called the zero-velocity curve (ZVC) (since for a point on it \( \vec{v} = 0 \)).
Epicycle Approximation I

We have defined the effective potential \( \Phi_{\text{eff}} = \Phi + \frac{L_z^2}{2R^2} \). This has a minimum at \((R, z) = (R_g, 0)\), where

\[
\frac{\partial \Phi_{\text{eff}}}{\partial R} = \frac{\partial \Phi}{\partial R} - \frac{L_z^2}{R^3} = 0
\]

The radius \( R = R_g \) corresponds to the radius of a circular orbit with energy

\[
E = \Phi(R_g, 0) + \frac{1}{2} v_c^2 = \Phi(R_g, 0) + \frac{L_z^2}{2R_g^2} = \Phi_{\text{eff}}.
\]

If we define \( x = R - R_g \) and expand \( \Phi_{\text{eff}} \) around the point \((x, z) = (0, 0)\) in a Taylor series we obtain

\[
\Phi_{\text{eff}} = \Phi_{\text{eff}}(R_g, 0) + (\Phi_x)x + (\Phi_z)z + (\Phi_{xz})xz + \frac{1}{2}(\Phi_{xx})x^2 + \frac{1}{2}(\Phi_{zz})z^2 + O(xz^2) + O(x^2z) + \text{etc}
\]

where

\[
\Phi_x = \left( \frac{\partial \Phi_{\text{eff}}}{\partial x} \right)(R_g, 0), \quad \Phi_{xx} = \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial x^2} \right)(R_g, 0), \quad \Phi_{xz} = \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial x \partial z} \right)(R_g, 0)
\]

By the definition of \( R_g \) and by symmetry considerations, we have that

\[
\Phi_x = \Phi_y = \Phi_{xz} = 0
\]
Epicycle Approximation II

In the epicycle approximation only terms up to second order are considered: all terms of order $xz^2$, $x^2z$ or higher are considered negligible. Defining

$$\kappa^2 \equiv \Phi_{xx} \quad \nu^2 \equiv \Phi_{zz}$$

we thus have that, in the epicycle approximation,

$$\Phi_{\text{eff}} = \Phi_{\text{eff}}(R_g, 0) + \frac{1}{2} \kappa^2 x^2 + \frac{1}{2} \nu^2 z^2$$

so that the equations of motion in the meridional plane become

$$\ddot{x} = -\kappa^2 x \quad \ddot{z} = -\nu^2 z$$

Thus, the $x$- and $z$-motions are simple harmonic oscillations with the epicycle frequency $\kappa$ and the vertical frequency $\nu$.

In addition, we have the circular frequency

$$\Omega(R) = \frac{v_c(R)}{R} = \sqrt{\frac{1}{R} \left( \frac{\partial \Phi}{\partial R} \right)_{(R,0)} = \frac{L_z}{R^2}}$$

that allows us to write

$$\kappa^2 = \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right)_{(R_g,0)} = \left( R \frac{d\Omega^2}{dR} + 4\Omega^2 \right)_{R_g}$$
Epicycle Approximation III

As we have seen before, for a realistic galactic potential \( \Omega < \kappa < 2\Omega \), where the limits correspond to a homogeneous mass distribution \((\kappa = 2\Omega)\) and the Kepler potential \((\kappa = \Omega)\).

In the epicycle approximation the motion is very simple:

\[
R(t) = A \cos(\kappa t + a) + R_g \\
z(t) = B \cos(\nu t + b) \\
\phi(t) = \Omega_g t + \phi_0 - \frac{2\Omega_g A}{\kappa R_g} \sin(\kappa t + a)
\]

with \(A, B, a, b,\) and \(\phi_0\) are all constants. The \(\phi\)-motion follows from

\[
\dot{\phi} = \frac{L_z}{R^2} = \frac{L_z}{R_g^2} \left(1 + \frac{x}{R_g}\right)^{-2} \simeq \Omega_g \left(1 - \frac{2x}{R_g}\right)
\]

Note that there are three frequencies \((\Omega, \kappa, \nu)\) and also three isolating integrals of motion in involution: \((E_R, E_z, L_z)\) with \(E_R = \frac{1}{2}(\dot{x}^2 + \kappa^2 x^2)\) and \(E_z = \frac{1}{2}(\dot{z}^2 + \nu^2 z^2)\) \(\triangleright\) all orbits are regular.

The motion in \((R, \phi)\) can be described as retrograde motion on an ellipse (the epicycle), whose guiding center (or epicenter) is in prograde motion around the center of the system.
An important question is: “When is the epicycle approximation valid?”

First consider the $z$-motion. The equation of motion, $\ddot{z} = -\nu^2 z$, implies a constant density in the $z$-direction. Hence, the epicycle approximation is only valid as long as $\rho(z)$ is roughly constant. This is only approximately true very close to the equatorial plane. In general, however, the epicycle approximation is poor for motion in the $z$-direction.

In the radial direction, we have to realize that the Taylor expansion is only accurate sufficiently close to $R = R_g$. Hence, the epicycle approximation is only valid for small librations around the guiding center, i.e. for orbits with an angular momentum that is close to that of the corresponding circular orbit.
A typical orbit in an axisymmetric potential. If the orbit admits two isolating integrals of motion, it would (ultimately) fill the entire area within the ZVC. Rather, the orbit is restricted to a sub-area within the ZVC, indicating that orbit admits a third isolating integral of motion.

Since this is not a classical integral of motion, and we don’t know how to express it in terms of the phase-space coordinates, it is simply called $I_3$.

Note that the point where orbit touches the ZVC can be used to ‘label’ $I_3$. The set $(E, L_z, I_3)$ uniquely defines an orbit.
The orbit shown on the previous page is a so-called short-axis tube orbit. This is the main orbit family in oblate potentials, and is associated with (parented by) circular orbits in the equatorial plane.

Orbits (c), (e) and (f) above are from the same orbit family. Orbits (a), (b) and (d) are special in that $L_z = 0$. 
Orbits in Axisymmetric Potentials V

Because of the centrifugal barrier only orbits with $L_z = 0$ are able to come arbitrarily close to the center.

However, not all orbits with $L_z = 0$ are box orbits. There is another family of zero-angular momentum orbits, namely two-dimensional loop orbits (e.g. orbit (d) on the previous slide). Their meridional plane is stationary (i.e. $\dot{\phi} = 0$) and their angular momentum vector is perpendicular to the $z$-axis. Hence, $I_3 = L$; note that $[L, L_z] = 0$.

Numerous authors have investigated orbits in axisymmetric potentials using numerical techniques. The main conclusions are:

- Most orbits in axisymmetric potentials designed to model elliptical galaxies are regular and appear to respect an effective third integral $I_3$.

- The principal orbit family in oblate potentials is the short-axis tube family, while two families of inner and outer long-axis tube orbits dominate in prolate potentials.

- In scale-free or cusped potentials several minor orbit families become important. These are boxlets associated with resonant parents.

- The fraction of phase-space occupied by stochastic, irregular orbits is generally (surprisingly) small.
Orbits in Triaxial Potentials I

Consider a triaxial density distribution with the major, intermediate, and minor axes aligned with the $x$, $y$, and $z$ axes, respectively.

In general, triaxial galaxies have four main orbit families: box orbits, and three tube orbits: short axis tubes, inner long-axis tubes, and outer long-axis tubes.

Orbit structure is different in the cusp, core, main body, and outer part (halo).

In the central core, the potential is harmonic; motion is that of a 3D harmonic oscillator. ▷ All orbits are box orbits parented by stable long-axial orbits.

Outside of the core region, frequencies become strongly radius (energy) dependent. There is an energy where $\omega_x = \omega_y$. At this 1 : 1-resonance the $y$-axial orbit becomes unstable and bifurcates into the short-axis tube family (two subfamilies with opposite senses of rotation).

At even higher energies, the $\omega_y : \omega_z = 1 : 1$ resonance makes the $z$-axial orbit unstable ▷ inner and outer long-axis tube families (each with two subfamilies with opposite senses of rotation).

At even larger radii (in the ‘halo’ of triaxial systems), the $x$-axial orbit becomes unstable ▷ box orbits are replaced by boxlets and stochastic orbits. The three families of tube orbits are also present.
Orbits in Triaxial Potentials II

- Box orbit
- Short-axis tube orbit
- Outer long-axis tube orbit
- Inner long-axis tube orbit
If the center is cusped rather than cored, resonant orbit families (boxlets) and stochastic orbits take over part of phase-space formerly held by box orbits. The extent to which this happens depends on the cusp slope.

Short-axis tubes contribute angular momentum in the $z$-direction and long-axis tubes contribute angular momentum in the $x$-direction $\triangleright$ the total angular momentum vector may point anywhere in the plane containing the long and short axes. NOTE: this can serve as a kinematic signature of triaxiality.

The closed loop orbit around the intermediate $y$-axis is unstable $\triangleright$ no family of intermediate-axis tubes.

Gas moves on closed, non-intersecting orbits. The only orbits with these properties are the stable loop orbits around the $x$- and $z$-axes. Consequently, gas and/or dust disks in triaxial galaxies can only exist in the $xy$-plane and the $yz$-plane, but not in the $xz$-plane. NOTE: these disks must be ellipsoidal rather than circular, and the velocity varies along ellipsoids.
Useful insight may be obtained from separable Stäckel models. These are the only known triaxial potentials that are completely integrable.

In Stäckel potentials all orbits are regular and part of one of the four main families.

Stäckel potentials are separable in ellipsoidal coordinates $(\lambda, \mu, \nu)$.

$(\lambda, \mu, \nu)$ are the roots for $\tau$ of

$$\frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} = 1$$

Here $\alpha < \beta < \gamma$ are constants and $-\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda$

Surfaces of constant $\lambda$ are ellipsoids.

Surfaces of constant $\mu$ are hyperboloids of one sheet.

Surfaces of constant $\nu$ are hyperboloids of two sheets.

Stäckel potentials are of the form:

$$\Phi(\vec{r}) = \Phi(\lambda, \mu, \nu) = -\frac{F_1(\lambda)}{(\lambda-\mu)(\lambda-\nu)} - \frac{F_2(\mu)}{(\mu-\nu)(\mu-\lambda)} - \frac{F_3(\nu)}{(\nu-\lambda)(\nu-\mu)}$$

with $F_1$, $F_2$ and $F_3$ arbitrary functions.
Stäckel Potentials II

The figure below shows contours of constant \((\lambda, \mu, \nu)\) plotted in the three planes (from left to right) \(xy\), \(xz\) and \(yz\).

At large distances, the ellipsoidal coordinates become close to spherical. Near the origin they are close to cartesian. For more details, see de Zeeuw (1985, MNRAS, 216, 273).

In triaxial Stäckel potentials all three integrals \((E, I_2, I_3)\) are analytic, and the orbits are confined by contours of constant ellipsoidal coordinates (see next page).

Although Stäckel potentials are a very special class, the fact that they are separable makes them ideally suited to gain insight. Most triaxial potentials that do not have a Stäckel form have orbital structures that are similar to those of Stäckel potentials.
Stäckel Potentials III

box orbits

inner long axis tube

outer long axis tube

short axis tube
### Orbits in Ellipsoidal Land; Summary

<table>
<thead>
<tr>
<th>System</th>
<th>Dim</th>
<th>Orbit Families</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oblate</td>
<td>$3D$</td>
<td>$S$</td>
</tr>
<tr>
<td>Prolate</td>
<td>$3D$</td>
<td>$I + O$</td>
</tr>
<tr>
<td>Triaxial</td>
<td>$3D$</td>
<td>$S + I + O + B$</td>
</tr>
<tr>
<td>Elliptic Disk</td>
<td>$2D$</td>
<td>$S + B$</td>
</tr>
</tbody>
</table>

*B* = box orbits  
*S* = short-axis tubes  
*I* = inner long-axis tubes  
*O* = outer long-axis tubes
Libration versus Rotation

Three-dimensional orbits

All tube orbits are built up from 2 librations and 1 rotation.
All box orbits are built up from 3 librations.
All boxlets are built up from 2 librations and 1 rotation.

Two-dimensional orbits

All loop orbits are built up from 1 libration and 1 rotation.
All box orbits are built up from 2 librations.
All boxlets are built up from 2 librations.
Rotating Potentials I

The figures of non-axisymmetric potentials may rotate with respect to inertial space.

The example of interest for astronomy are barred potentials, which are rotating with a certain pattern speed.

We express the pattern speed in angular velocity $\Omega_p = \Omega_p \hat{e}_z$

In what follows we denote by $\frac{d\bar{a}}{dt}$ the rate of change of a vector $\bar{a}$ as measured by an inertial observer, and by $\dot{\bar{a}}$ the rate of change as measured by an observer corotating with the figure.

It is straightforward to show that

$$\frac{d\bar{a}}{dt} = \dot{\bar{a}} + \Omega_p \times \bar{a}$$

Applying this twice to the position vector $\bar{r}$, we obtain

$$\frac{d^2\bar{r}}{dt^2} = \frac{d}{dt} \left( \dot{\bar{r}} + \Omega_p \times \bar{r} \right)$$

$$= \ddot{\bar{r}} + \Omega_p \times \dot{\bar{r}} + \Omega_p \times \frac{d\bar{r}}{dt}$$

$$= \ddot{\bar{r}} + \Omega_p \times \dot{\bar{r}} + \Omega_p \times \left( \dot{\bar{r}} + \Omega_p \times \bar{r} \right)$$

$$= \ddot{\bar{r}} + 2 \left( \Omega_p \times \dot{\bar{r}} \right) + \Omega_p \times \left( \Omega_p \times \bar{r} \right)$$
Rotating Potentials II

Since Newton’s laws apply to inertial frames we have that

\[ \ddot{\vec{r}} = -\vec{\nabla} \Phi - 2 \left( \vec{\Omega}_p \times \dot{\vec{r}} \right) - \vec{\Omega}_p \times \left( \vec{\Omega}_p \times \vec{r} \right) \]

Note the two extra terms: \(-2 \left( \vec{\Omega}_p \times \dot{\vec{r}} \right)\) represents the Coriolis force and 
\(-\vec{\Omega}_p \times \left( \vec{\Omega}_p \times \vec{r} \right)\) the centrifugal force.

The energy is given by

\[ E = \frac{1}{2} \left( \frac{d\vec{r}}{dt} \right)^2 + \Phi(\vec{r}) \]
\[ = \frac{1}{2} \left( \dot{\vec{r}} + \vec{\Omega}_p \times \vec{r} \right)^2 + \Phi(\vec{r}) \]
\[ = \frac{1}{2} \ddot{\vec{r}}^2 + \dot{\vec{r}} \cdot \left( \vec{\Omega}_p \times \vec{r} \right) + \frac{1}{2} \left( \vec{\Omega}_p \times \vec{r} \right)^2 + \Phi(\vec{r}) \]
\[ = \frac{1}{2} \ddot{\vec{r}}^2 + \dot{\vec{r}} \cdot \left( \vec{\Omega}_p \times \vec{r} \right) + \frac{1}{2} |\vec{\Omega}_p \times \vec{r}|^2 + \Phi(\vec{r}) \]
\[ = E_J + \dot{\vec{r}} \cdot \left( \vec{\Omega}_p \times \vec{r} \right) + |\vec{\Omega}_p \times \vec{r}|^2 \]

Where we have defined Jacobi’s Integral

\[ E_J \equiv \frac{1}{2} \ddot{\vec{r}}^2 + \Phi(\vec{r}) - \frac{1}{2} |\vec{\Omega}_p \times \vec{r}|^2 \]
The importance of $E_J$ becomes apparent from the following:

\[
\frac{dE_J}{dt} = \vec{r} \cdot \frac{d}{dt} \left( \frac{d}{dt} \right) - (\vec{\Omega}_p \times \vec{r}) \cdot \frac{d}{dt} (\vec{\Omega}_p \times \vec{r})
\]

\[
= \vec{r} \left[ \vec{\ddot{r}} + (\vec{\Omega}_p \times \vec{\dot{r}}) \right] + \vec{\nabla} \Phi \cdot \vec{\dot{r}} - (\vec{\Omega}_p \times \vec{r}) \cdot (\vec{\Omega}_p \times \vec{\dot{r}})
\]

Here we have used that $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$ and that $d\vec{\Omega}_p / dt = 0$. If we multiply the equation of motion with $\vec{\dot{r}}$ we obtain that

\[
\dot{r} \cdot \ddot{r} + \dot{r} \cdot \vec{\nabla} \Phi + 2 \dot{r} \cdot (\vec{\Omega}_p \times \vec{r}) + \dot{r} \cdot \left[ \vec{\Omega}_p \times (\vec{\Omega}_p \times \vec{r}) \right] = 0
\]

\[\Leftrightarrow \dot{r} \cdot \ddot{r} + \dot{r} \cdot \vec{\nabla} \Phi + (\vec{\Omega}_p \times \vec{r}) \cdot (\vec{r} \times \vec{\Omega}_p) = 0\]

Where we have used that $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$. Since $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ we have that

\[
\frac{dE_J}{dt} = 0
\]

The **Jacobi Integral** is a conserved quantity, i.e. an integral of motion.
For comparison, since $\Phi = \Phi(t)$ the Hamiltonian is explicitly time-dependent; consequently, the total energy $E$ is not a conserved quantity (i.e. it is not an integral of motion).

The angular momentum is given by

$$\vec{L} = \vec{r} \times \frac{d\vec{r}}{dt} = \vec{r} \times \dot{\vec{r}} + \vec{r} \times (\vec{\Omega}_p \times \vec{r})$$

This allows us to write

$$\vec{\Omega}_p \cdot \vec{L} = \vec{\Omega}_p \cdot (\vec{r} \times \dot{\vec{r}}) + \vec{\Omega}_p \cdot \left[ \vec{r} \times (\vec{\Omega}_p \times \vec{r}) \right] = \dot{\vec{r}} \cdot (\vec{\Omega}_p \times \vec{r}) + |\vec{\Omega}_p \times \vec{r}|^2$$

from which we obtain that

$$E_J = E - \vec{\Omega}_p \cdot \vec{L}$$

Thus in a rotating, non-axisymmetric potential neither the energy $E$ nor the angular momentum $L$ are conserved but the Jacobi integral is conserved.

Note that $E_J$ is the sum of $\frac{1}{2} \dot{\vec{r}}^2 + \Phi$, which would be the energy if the frame were not rotating, and the quantity $-\frac{1}{2} |\vec{\Omega}_p \times \vec{r}|^2 = -\frac{1}{2} \Omega_p^2 R^2$, which can be thought of as the potential energy corresponding to the centrifugal force.
If we now define the **effective potential**

\[ \Phi_{\text{eff}} = \Phi - \frac{1}{2} \Omega_p^2 R^2 \]

the equation of motion becomes

\[ \ddot{\vec{r}} = -\vec{\nabla} \Phi_{\text{eff}} - 2(\vec{\Omega}_b \times \dot{\vec{r}}) \]

and the **Jacobi integral** is

\[ E_J = \frac{1}{2} |\dot{\vec{r}}|^2 + \Phi_{\text{eff}} \]

An orbit with a given value for it’s **Jacobi Integral** is restricted in its motion to regions where \( E_J \leq \Phi_{\text{eff}} \). The surface \( \Phi_{\text{eff}} = E_J \) is, therefore, often called the **zero-velocity surface**.

The **effective potential** has five points where both \( \partial \Phi_{\text{eff}} / \partial x \) and \( \partial \Phi_{\text{eff}} / \partial y \) vanish. These points, \( L_1 \) to \( L_5 \), are called the **Lagrange Points** (cf. the restricted three-body problem).

- Motion around \( L_3 \) (minimum of \( \Phi_{\text{eff}} \)) is always **stable**.
- Motion around \( L_1 \) and \( L_2 \) (saddle points of \( \Phi_{\text{eff}} \)) is always **unstable**.
- Motion around \( L_4 \) and \( L_5 \) (maxima of \( \Phi_{\text{eff}} \)) can be **stable** or **unstable** depending on the potential.

**NOTE:** **stable/unstable** refers to whether orbits remain close to Lagrange points or not.
Illustration of Lagrange points ($L_1$ to $L_5$) in the Sun-Earth-Moon system.
Illustration of Lagrange points \((L_1 \text{ to } L_5)\) in a logarithmic potential. The annulus bounded by the circle through \(L_1, L_2\) and \(L_3, L_4\) (in red) is called the region of corotation.
Lindblad Resonances I

Let \((R, \theta)\) be the polar coordinates that are corotating with the planar potential \(\Phi(R, \theta)\). If the non-axisymmetric distortions of the potential, which has a pattern speed \(\Omega_p\), are sufficiently small then we may write

\[
\Phi(R, \theta) = \Phi_0(R) + \Phi_1(R, \theta) \quad \mid \Phi_1/\Phi_0 \mid \ll 1
\]

It is useful to consider the following form for \(\Phi_1\)

\[
\Phi_1(R, \theta) = \Phi_p(R) \cos(m\theta)
\]

where \(m = 2\) corresponds to a (weak) bar.

In the epicycle approximation the motion in \(\Phi_0(R)\) is that of an epicycle, with frequency \(\kappa(R)\), around a guiding center that rotates with frequency

\[
\Omega(R) = \sqrt{\frac{1}{R} \frac{d\Phi_0}{dR}}.
\]

In the presence of \(\Phi_1(R, \theta)\), the movement of the guiding center is \(\theta_0(t) = [\Omega(R) - \Omega_p] t\). In addition to the natural frequencies \(\Omega(R)\) and \(\kappa(R)\) there is a new frequency \(\Omega_p\). Because \(\Phi_1(R, \theta)\) has \(m\)-fold symmetry, the guiding center at \(R\) finds itself at effectively the same location in the \((R, \theta)\)-plane with a frequency \(m [\Omega(R) - \Omega_p]\).
Motion in the $R$-direction becomes that of a harmonic oscillator of natural frequency $\kappa(R)$ that is driven by the frequency $m [\Omega(R) - \Omega_p]$.

At several $R$ the natural and driving frequencies are in resonance.

1. **Corotation:** $\Omega(R) = \Omega_p$
   (The guiding center corotates with the potential).

2. **Lindblad Resonances:** $m [\Omega(R) - \Omega_p] = \pm \kappa(R)$
   Most important of these are:
   
   - $\Omega(R) - \frac{\kappa}{2} = \Omega_p$ : Inner Lindblad Resonance.
   - $\Omega(R) + \frac{\kappa}{2} = \Omega_p$ : Outer Lindblad Resonance.
   - $\Omega(R) - \frac{\kappa}{4} = \Omega_p$ : Ultra Harmonic Resonance.

Depending on $\Phi(R, \theta)$ and $\Omega_p$ one can have 0, 1, or 2 ILRs. If there are two, we distinguish between the Inner Inner Lindblad Resonance (IILR) and the Outer Inner Lindblad Resonance (OILR).

If a cusp (or BH) is present there is always 1 ILR, because $\Omega(R) - \kappa(R)/2$ increases monotonically with decreasing $R$. 

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Lindblad Resonances III

Lindblad Resonances play an important role for orbits in barred potentials.
As an example, we discuss the orbital families in a planar, rotating, logarithmic potential.

(a) Long-axial orbit $\rightarrow$ stable, oval, prograde, and oriented $\parallel$ to $\Phi_{\text{eff}}$. ($x_1$-family).

(b) Short-axial orbit $\rightarrow$ stable, oval, retrograde, and oriented $\perp$ to $\Phi_{\text{eff}}$. At $E > E_1$ (at IILR), family (b) becomes unstable and bifurcates into two prograde loop families that are oriented perpendicular to $\Phi_{\text{eff}}$. The stable (unstable) family is called the $x_2$ ($x_3$) family. At the same energy the $x_1$-orbits develop self-intersecting loops.

At $E > E_2$ (at OILR) the $x_2$ and $x_3$ families disappear. The $x_1$ family looses its self-intersecting loops.

In the vicinity of corotation there are families of orbits around $L_4$ and $L_5$ (if these are stable).

At large radii beyond CR $\Omega_p \gg \Omega(R)$. Consequently, the orbits effectively see a circular potential and the orbits become close to circular rosettes.